Calculus-I

Evolutes and Involutes

Curvature

Measure of bending is called curvature. It is denoted by K.

Radius of curvature is the reciprocal of curvature. It is denoted by ρ.

If equation of curve is:

- $\boldsymbol{s = f(\psi)}: \rho = \frac{ds}{d\psi}$

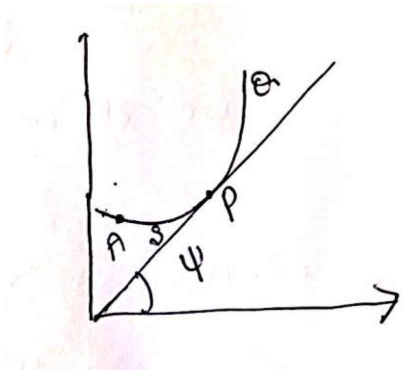
- $\boldsymbol{y = f(x)\ i.e. in\ Cartesian\ form}: \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

- $\boldsymbol{x = f(t)\ and\ y = g(t)\ i.e.\ in\ Parametric\ form}: \rho = \frac{\left(\left[\frac{dx}{dt}\right]^2+\left[\frac{dy}{dt}\right]^2\right)^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}$

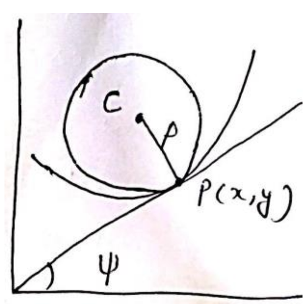
- $\boldsymbol{r = f(\theta)\ i.e. in\ polar\ form}: \rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$

- $\boldsymbol{f(x,y) = c\ i.e. in\ Implicit\ form}: \rho = \frac{(f_x^2+f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2+f_{yy}f_x^2-2f_xf_yf_{xy}}$

- ***Curvature of a curve at a point***: Let P and Q be two neighbouring points on a curve. Let arc PQ = s, where A is a fixed point on curve. Let ψ be angle made by tangent at P with x-axis. Then $\frac{d\psi}{ds}$ is called curvature of curve at point P.Thus curvature is defined as rate of turning of tangent w.r.t. arc length.



<u>Centre of Curvature and Circle of Curvature</u>: Let $P(x, y)$ be any point on the curve $y = f(x)$. Let ψ be angle made by tangent at P with x-axis. Let C be point on positive direction of normal to the curve at point P such that $CP = \rho$. Then C is called centre of curvature to the curve at point P. The circle with centre C and radius ρ is called circle of curvature at P.



Centre of curvature for the point (x, y) is $[\bar{x}, \bar{y}]$ where

$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$ and $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$, $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

Evolutes and Involutes

The locus of centre of curvature of a curve is called evolute and the given curve is called involute.

1. Find centre of curvature at any point (x, y) of the parabola $y^2 = 4ax$. also find its evolutes.

Sol. The given curve is $y^2 = 4ax$. Differentiating both sides w.r.t. x, we get

$$2yy_1 = 4a \xrightarrow{yields} y_1 = \frac{2a}{y}$$

Again Differentiating both sides w.r.t. x, we get

$$y_2 = \frac{d^2y}{dx^2} = \frac{d\left(\frac{2a}{y}\right)}{dx} = \frac{-2a}{y^2}\frac{dy}{dx} = \frac{-2a}{y^2}.\frac{2a}{y} = \frac{-4a^2}{y^3}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{\frac{-4a^2}{y^3}} = x + \frac{(y^2+4a^2)}{2a} = \frac{2ax+4ax+4a^2}{2a} = 3x + 2a$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = y + \frac{\left(1+\frac{4a^2}{y^2}\right)}{\frac{-4a^2}{y^3}} = y - \frac{y(y^2+4a^2)}{4a^2} = \frac{-y^3}{4a^2} = \frac{-(4ax)^{3/2}}{4a^2} = \frac{-2x^{3/2}}{\sqrt{a}}$$

∴ Centre of curvature is $\left(3x + 2a, \ \frac{-2x^{3/2}}{\sqrt{a}}\right)$.

To find evolute eliminate x from $\bar{x}$ and $\bar{y}$.

$\bar{x} = 3x + 2a \xrightarrow{yields} x = \frac{\bar{x}-2a}{3}$ and

$\bar{y} = \frac{-2x^{3/2}}{\sqrt{a}} \xrightarrow{yields} = \bar{y}^2 = \frac{4x^3}{a} \xrightarrow{yields} \bar{y}^2 = \frac{4\left(\frac{\bar{x}-2a}{3}\right)^3}{a} \xrightarrow{yields} 27a\bar{y}^2 = 4(\bar{x} - 2a)^3$ is the locus of $[\bar{x}, \bar{y}]$

∴ Evolute is $27a\bar{y}^2 = 4(\bar{x} - 2a)^3$.

2. Find evolutes of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. The given curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Differentiating both sides w.r.t. x, we get

$$\frac{2x}{a^2}+\frac{2yy_1}{b^2}=1\xrightarrow{yields}y_1=\frac{-b^2x}{a^2y}$$

And $\quad y_2=\frac{d^2y}{dx^2}=\frac{d\left(\frac{-b^2x}{a^2y}\right)}{dx}=\frac{-b^2}{a^2}\frac{(y-xy_1)}{y^2}=\frac{-b^4}{a^2y}$

$$\bar{x}=x-\frac{y_1(1+y_1^2)}{y_2}=x-\frac{\frac{-b^2x}{a^2y}\left(1+\frac{b^4x^2}{a^4y^2}\right)}{\frac{-b^4}{a^2y}}=x-\frac{x(a^4y^2+b^4x^2)}{a^4b^2}=x-\frac{xy^2}{b^2}-\frac{x^3b^2}{a^4}$$

$$=x-x\left(1-\frac{x^2}{a^2}\right)-\frac{x^3b^2}{a^4}=\frac{x^3}{a^2}-\frac{x^3b^2}{a^4}=\frac{(a^2-b^2)x^3}{a^4}$$

$$\bar{y}=y+\frac{(1+y_1^2)}{y_2}=y+\frac{\left(1+\frac{b^4x^2}{a^4y^2}\right)}{\frac{-b^4}{a^2y}}=y-\frac{y(a^4y^2+b^4x^2)}{a^2b^4}=y-\frac{a^2y^3}{b^4}-\frac{yx^2}{a^2}$$

$$=y-\frac{a^2y^3}{b^4}-y\left(1-\frac{y^2}{b^2}\right)=\frac{y^3}{b^2}-\frac{a^2y^3}{b^4}=\frac{(b^2-a^2)y^3}{b^4}$$

$\therefore$ Centre of curvature is $\left(\frac{(a^2-b^2)x^3}{a^4},\ \frac{(b^2-a^2)y^3}{b^4}\right)$.

$$\bar{x}=\frac{(a^2-b^2)x^3}{a^4}\xrightarrow{yields}a\bar{x}=\frac{(a^2-b^2)x^3}{a^3}\xrightarrow{yields}(a\bar{x})^2=\left[(a^2-b^2)\frac{x^3}{a^3}\right]^2$$

$$\xrightarrow{yields}(a\bar{x})^2=(a^2-b^2)^2\left(\frac{x}{a}\right)^6$$

and

$$\bar{y}=\frac{(b^2-a^2)y^3}{b^4}\xrightarrow{yields}=b\bar{y}=\frac{(b^2-a^2)y^3}{b^3}\xrightarrow{yields}(b\bar{y})^2=\left[\frac{(b^2-a^2)y^3}{b^3}\right]^2$$

$$\xrightarrow{yields}(b\bar{y})^2=(b^2-a^2)^2\left(\frac{y}{b}\right)^6\xrightarrow{yields}(b\bar{y})^2=(a^2-b^2)^2\left(\frac{y}{b}\right)^6$$

Now $(a\bar{x})^{2/3}+(b\bar{y})^{2/3}=(a^2-b^2)^{2/3}\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right]$

$$\xrightarrow{yields}(a\bar{x})^{2/3}+(b\bar{y})^{2/3}=(a^2-b^2)^{2/3}$$

is the locus of $[\bar{x},\bar{y}]$

$\therefore$ Evolute is $(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}$

3. If $\rho_{1,}$ ρ_2 be the radii of curvature at the extremities of a focal chord of a parabola , where semi latus rectum is l. Prove that $(\rho_1)^{-\frac{2}{3}} + (\rho_2)^{-\frac{2}{3}} = (l)^{-\frac{2}{3}}$

Sol. $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

$y^2 = 4ax$ Differentiating both sides w.r.t. x, we get

$$2yy_1 = 4a \xrightarrow{yields} y_1 = \frac{2a}{y} \xrightarrow{yields} y_1 = \frac{2a}{2\sqrt{ax}} \xrightarrow{yields} y_1 = \sqrt{\frac{a}{x}}$$

Again Differentiating both sides w.r.t. x, we get

$y_2 = \frac{-\sqrt{a}}{2x^{\frac{3}{2}}}$

Therefore $\rho = \frac{\left(1+\frac{a}{x}\right)^{\frac{3}{2}}}{\frac{-\sqrt{a}}{2x^{\frac{3}{2}}}} = \frac{-2(x+a)^{\frac{3}{2}}}{\sqrt{a}} \xrightarrow{yields} \rho^2 = \frac{4(x+a)^3}{a}$..................(1)

Let P($at^2, 2at$) be any point on one end of focal chord of the parabola then Q$\left(\frac{a}{t^2}, \frac{2a}{t}\right)$ be the point on other end of focal chord.

From eqn(1) $\quad \rho_1^2 = \frac{4(at^2 + a)^3}{a}$

$\rho_1^2 = 4a^2\left(t^2+1\right)^3$,.........(2) for ρ_2 replace t to 1/t ,

$$\rho_2^2 = 4a^2\left(\frac{1}{t^2}+1\right)^3 \text{,.........(3)}$$

From eqns (2) and (3), we $\rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}$

4. Find radius of curvature at (r, θ) on the curve $r^n = a^n \cos n\theta$

Sol. Taking log on both sides, we get $nlogr = nloga + \log\cos n\theta$

Differentiating both sides w. r. t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} \xrightarrow{yields} r_1 = -r\tan n\theta$$

Differentiating both sides w. r. t. θ, we get

$$r_2 = -r_1 \tan n\theta - rn\, sec^2 n\theta = r tan^2 n\theta - rn\, sec^2 n\theta$$

$$\rho = \frac{(r^2 + {r_1}^2)^{3/2}}{r^2 + 2{r_1}^2 - rr_2} = \frac{(r^2 + r^2 tan^2 n\theta)^{3/2}}{r^2 + 2r^2 tan^2 n\theta + r^2(n\, sec^2 n\theta - tan^2 n\theta)}$$

$$= \frac{r^3 sec^3 n\theta}{r^2[1 + 2tan^2 n\theta + (n\, sec^2 n\theta - tan^2 n\theta)]}$$

$$= \frac{rsec^3 n\theta}{[1 + tan^2 n\theta + n\, sec^2 n\theta]} = \frac{rsec^3 n\theta}{[sec^2 n\theta + n\, sec^2 n\theta]} = \frac{rsecn\theta}{[1+n\,]} = \frac{a^n}{[1+n\,]r^{n-1}}$$

5. Given parabola $y^2 = 4ax$. If P(x,y) be any point on the parabola and S(a,0) is the focus show that ρ^2 varies as $(SP)^3$.

Sol. $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

$y^2 = 4ax$ Differentiating both sides w.r.t. x, we get

$$2yy_1 = 4a \xrightarrow{yields} y_1 = \frac{2a}{y} \xrightarrow{yields} y_1 = \frac{2a}{2\sqrt{ax}} \xrightarrow{yields} y_1 = \sqrt{\frac{a}{x}}$$

Again Differentiating both sides w.r.t. x, we get

$y_2 = \frac{-\sqrt{a}}{2x^{\frac{3}{2}}}$

Therefore $\rho = \frac{\left(1+\frac{a}{x}\right)^{\frac{3}{2}}}{\frac{-\sqrt{a}}{2x^{\frac{3}{2}}}} = \frac{-2(x+a)^{\frac{3}{2}}}{\sqrt{a}} \xrightarrow{yields} \rho^2 = \frac{4(x+a)^3}{a}$

Let P(x,y) be any point on the parabola and S(a,0) is the focus.

Then (SP)=$\sqrt{(x-a)^2 + (y-0)^2} = \sqrt{x^2 + a^2 - 2ax + 4ax} = x + a$

$\xrightarrow{yields} (SP)^3 = (x+a)^3$

Therefore $\rho^2 = \frac{4(x+a)^3}{a} = \frac{4(SP)^3}{a}$

Hence ρ^2varies as $(SP)^3$.

6. Find the radius of curvature of $s = a\ \log(\sec\psi + \tan\psi) + a\sec\psi\tan\psi$

Sol. $s = a\ \log(\sec\psi + \tan\psi) + a\sec\psi\tan\psi$

The required radius of curvature is $= \rho = \frac{ds}{d\psi}$

$$\frac{ds}{d\psi} = \frac{a}{\sec\psi + \tan\psi}[\sec\psi\tan\psi + (\sec\psi)^2] + a[\sec\psi\,(\sec\psi)^2 + \tan\psi\sec\psi\tan\psi]$$

$$\frac{ds}{d\psi} = \text{a}\sec\psi + a\sec\psi\ \ [(\sec\psi)^2 + (\tan\psi)^2]$$

$$\frac{ds}{d\psi} = a\sec\psi\ \ [1 + (\sec\psi)^2 + (\tan\psi)^2]$$

$$\frac{ds}{d\psi} = a\sec\psi\ \ [2(\sec\psi)^2] = 2a(\sec\psi)^3$$

7. Find the radius of curvature of $x = a(\cos t + t\sin t\ ), y = a(\sin t\ - t\cos t)$

Sol. $\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)\ ,\ \frac{dy}{dt} = a(\cos t - \cos t + t\sin t)$

$$\frac{dx}{dt} = a\ t\cos t\ \ ,\frac{dy}{dt} = a\ t\sin t$$

$$\frac{d^2x}{dt^2} = a(\cos t - t\sin t)\ , \qquad \frac{d^2y}{dt^2} = a(\sin t + t\cos t)$$

Radius of curvature at any point of the given curve $= \rho = \frac{\left(\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2\right)^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}$

$$\rho = \frac{(a^2t^2)^{\frac{3}{2}}}{(a\ t\cos t)\big(a(\sin t + t\cos t)\big) - (a\ t\sin t)(a(\cos t - t\sin t))}$$

$$\rho = \frac{(at)^3}{a^2[(t\cos t)((\sin t + t\cos t) - (t\sin t)(\cos t - t\sin t)]} = \frac{a^3t^3}{a^2t^2} = at$$

8. Find radius of curvature at origin to the curve $y^4 + x^3 + a(x^2 + y^2) - a^2y = 0$

Equation of tangent to the curve at origin is $-a^2y = 0$ i.e. $y = 0$ i.e. x-axis.

By Newton's method $\rho = \lim_{x\to 0}\frac{x^2}{2y}$

Divide the equation by 2y, $\left(\frac{y^3}{2} + \frac{x^3}{2y} + a\left(\frac{x^2}{2y} + \frac{y}{2}\right) - \frac{a^2}{2}\right) = 0$

Take $x \to 0\ (then\ y \to 0)\ and\ \ \frac{x^2}{2y} \to \rho$

$0 + 0 + a(\rho + 0) = \frac{a^2}{2} \xrightarrow{yields} \rho = \frac{a}{2}$

9. If ρ_1, ρ_2 be the radii of curvature at the extremities of the conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that $\left(\rho_1{}^{2/3}+\rho_2{}^{2/3}\right)(ab)^{2/3}=a^2+b^2$

Solution: Let P(a $\cos\theta, b\sin\theta$) be any point on the ellipse. Then the coordinates of the extremity of the conjugate diameter will be Q $(-a\ sin\theta, b\cos\theta)$

Radius of the curvature of the ellipse at P (x, y) is

$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$$

For ρ_1 replace (x, y) by (a $\cos\theta, b\sin\theta$)

$$\rho_1=\frac{(a^4b^2sin^2\theta+b^4a^2cos^2\theta)^{\frac{3}{2}}}{a^4b^4}$$

For ρ_2 replace (x, y) by $(-a\ sin\theta, b\cos\theta)$

$$\rho_2=\frac{(a^4b^2cos^2\theta+b^4a^2sin^2\theta)^{\frac{3}{2}}}{a^4b^4}$$

$$\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}}=\frac{a^4b^2sin^2\theta+b^4a^2cos^2\theta}{a^4b^{4\frac{2}{3}}}+\frac{a^4b^2cos^2\theta+b^4a^2sin^2\theta}{a^4b^{4\frac{2}{3}}}$$

$$\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}}=\frac{a^2+b^2}{(ab)^{\frac{2}{3}}}$$

$$\left(\rho_1{}^{2/3}+\rho_2{}^{2/3}\right)(ab)^{2/3}=a^2+b^2$$

10. Find radius of curvature of the curve $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$.

Solution: $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$

$$\frac{dx}{d\theta}=a(1-cos\theta)\frac{dy}{d\theta}=a\ sin\theta$$

$$\frac{d^2x}{d\theta^2}=sin\theta\frac{d^2y}{d\theta^2}=a\ cos\theta$$

$$\rho=\frac{\left[\frac{dx^2}{d\theta}+\frac{dy^2}{d\theta}\right]^{\frac{3}{2}}}{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2}-\frac{d^2x}{d\theta^2}\frac{dy}{d\theta}}$$

$$\rho=\frac{a\ (2-2cos\theta)^{\frac{3}{2}}}{cos\theta-1}$$

$$\rho=4asin\theta/2$$

11. Show that the ellipse $x = a\cos t$, $y = b\sin t$, $a > 0, b > 0$ has its largest curvature on the major axis and smallest curvature on the minor axis.

Solution: $x = a\cos t$ $y = b\sin t$

$$x' = -a\,sint \qquad y' = b\cos t$$
$$x'' = -a\,cost \qquad y'' = -b\,sint$$

Curvature K = $\frac{x'y''-x''y'}{(x'^2+y'^2)^{\frac{3}{2}}}$

K = $\frac{2\sqrt{2}ab}{(a^2+b^2-(a^2+b^2)cos2t)^{\frac{3}{2}}}$

(i) K will be maximum when cos 2t is maximum and max cos 2t =1

Cos 2t = cos 0, cos 2π

$$\therefore \quad t = 0, \quad \pi$$
$$x = \pm a, y = 0$$

K is maximum at (a, 0) and (-a, 0)

i.e K is maximum on its major axis

(ii) K will be minimum when cos 2t is minimum and minimum cos 2t = -1

Cos 2t = cosπ and cos -π

$$\therefore \quad t = \frac{\pi}{2}, -\frac{\pi}{2}$$

i.e K is minimum on its minor axis

Definite Integrals

Important Properties of Definite Integrals

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b 1dx = b - a$

3. $$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

4. Change of variable of integration is immaterial as long as the limits of integration remain the same, i.e.

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

5. If the limits are interchanged, i.e. the upper limit becomes the lower limit and vice versa, then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

6. If f is a piecewise continuous function, then the integral is broken at the points of discontinuity or at the points where the definition of f changes,

i.e. $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

7. $$\int_{-a}^a f(x)dx = \int_0^a (f(x) + f(-x))dx$$

$$= 0 iff(x) isodd$$

$$= 2\int_0^a f(x)dx iff(x) iseven$$

8. $$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Another result that can be derived from this property is

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

9. $$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= 0 \quad if f(2a - x) = -f(x) \qquad = 2\int_0^a f(x)dx \quad if f(2a-x) = f(x)$$

Q1 Integrate the following integral:

$$\int_{\frac{\pi}{2}}^{\pi} \left(\frac{1-\sin x}{1-\cos x}\right) dx$$

Ans. Let I = $\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) dx$$

$$= \left(-e^x \cot\frac{x}{2}\right)_{\frac{\pi}{2}}^{\pi}$$

$$\left[\because \int e^x \left(f(x) + f'(x)\right) dx = e^x f(x)\right]$$

$$= -e^{\pi}\cot\frac{\pi}{2} - \left(-e^{\frac{\pi}{2}}\cot\frac{\pi}{4}\right)$$

$= -e^{\pi(0)} + e^{\frac{\pi}{2}}(1) = e^{\frac{\pi}{2}}$

Q2 Integrate the following integral:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$$

Ans. Let I = $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$

$\Rightarrow$ I = $\int_0^{\frac{\pi}{4}} \frac{\frac{\sin x \cos x}{\cos x \cos x \cos^2 x}}{1 + \frac{\sin^4 x}{\cos^4 x}}\, dx$

[Dividing each term by x^4]

$\Rightarrow$ I = $\int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x}\, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{2 \tan x \sec^2 x}{1 + \tan^4 x}\, dx$ ……….(i)

Putting $\tan^2 x = t$

$\Rightarrow$ $2 \tan x \frac{d}{dx}(\tan x) = \frac{dt}{dx}$

$\Rightarrow$ $2 \tan x \sec^2 x\, dx = dt$

Limits of integration when $x = 0, t = \tan^2 x = \tan^2 0° = 0$ and when $x = \frac{\pi}{4}, t = \tan^2 \frac{\pi}{4} = 1$

$\therefore$ I = $\frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$

= $\frac{1}{2}\left(\tan^{-1} t\right)_0^1$

= $\frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1} 0\right)$

= $\frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$

Q3 Integrate the following integral:

$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$

Ans. Let I = $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

= $\int_0^{\frac{\pi}{2}} \frac{1}{4}\left(3\sin x - \sin 3x\right)\, dx$

= $\frac{1}{4}\left[3\left(-\cos x\right) - \left(\frac{-\cos 3x}{3}\right)\right]_0^{\frac{\pi}{2}}$

$\Rightarrow$ I = $\frac{1}{4}\left[-3\cos x - \frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{2}}$

= $\frac{1}{4}\left[\left(-3\cos\frac{\pi}{2} + \frac{1}{3}\cos\frac{3\pi}{2}\right) - \left(-3\cos 0 + \frac{1}{3}\cos 0\right)\right]$

$\Rightarrow$ I = $\frac{1}{4}\left[-3\times 0 + \frac{1}{3}\times 0 + 3\times 1 - \frac{1}{3}\times 1\right]$

$= \frac{1}{4}\left(3 - \frac{1}{3}\right)$

$= \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$

Q4 Integrate the following integral:

$$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

Ans. Let I = $\int_0^1 \frac{1}{1+x-\sqrt{x}}\, dx$

$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{\left(\sqrt{1+x}+\sqrt{x}\right)\left(\sqrt{1+x}-\sqrt{x}\right)}\, dx$

$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}\, dx$

$\Rightarrow$ I = $\int_0^1 \left(\sqrt{1+x}+\sqrt{x}\right)\, dx$

$= \int_0^1 (1+x)^{\frac{1}{2}}\, dx + \int_0^1 (x)^{\frac{1}{2}}\, dx$

$= \frac{\left[(1+x)^{\frac{3}{2}}\right]_0^1}{\frac{3}{2}(1)} + \frac{\left[(x)^{\frac{3}{2}}\right]_0^1}{\frac{3}{2}(1)}$

$\Rightarrow$ I = $\frac{2}{3}\left[(2)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] + \frac{2}{3}\left[(1)^{\frac{3}{2}} - 0\right]$

$= \frac{2}{3}\left[2\sqrt{2}-1\right]+\frac{2}{3}[1-0]$

$\Rightarrow$ I = $\frac{4\sqrt{2}}{3}-\frac{2}{3}+\frac{2}{3}=\frac{4\sqrt{2}}{3}$

. $\int_1^4 \left[|x-1|+|x-2|+|x-3|\right] dx$

Q5 Integrate the following integral:

Ans. Let I = $\int_1^4 \left(|x-1|+|x-2|+|x-3|\right) dx$ ……….(i)

If $x-1=0, x-2=0, x-3=0$ we get $x=1, x=2, x=3$

$\Rightarrow$ $x=2,3(1,4)$

$\therefore$ I = $\int_1^2 \left(|x-1|+|x-2|+|x-3|\right) dx+\int_2^3 \left(|x-1|+|x-2|+|x-3|\right) dx+\int_3^4 \left(|x-1|+|x-2|+|x-3|\right) dx$

= $\int_1^2 \left\{(x-1)-(x-2)-(x-3)\right\} dx+\int_2^3 \left\{(x-1)+(x-2)-(x-3)\right\} dx+\int_3^4 \left\{(x-1)+(x-2)+(x-3)\right\} dx$

$\Rightarrow$ I = $\int_1^2 (x-1-x+2-x+3)\ dx+\int_2^3 (x-1+x-2-x+3)\ dx+\int_3^4 (x-1+x-2+x-3)\ dx$

$\Rightarrow$ I = $\int_1^2 (4-x)\ dx+\int_2^3 (x)\ dx+\int_3^4 (3x-6)\ dx$

$= \left(4x-\frac{x^2}{2}\right)_1^2+\left(\frac{x^2}{2}\right)_2^3+\left(\frac{3x^2}{2}-6x\right)_3^4$

$\Rightarrow$ I = $(8-2)-\left(4-\frac{1}{2}\right)+\frac{9}{2}-\frac{4}{2}+(24-24)\left(\frac{27}{2}-18\right)$

$\Rightarrow$ I = $6-4+\frac{1}{2}+\frac{5}{2}-\left(-\frac{9}{2}\right)$

= $2+\frac{1}{2}+\frac{5}{2}+\frac{9}{2}$ = $\frac{19}{2}$

Q6: Integrate the following integral:

$$\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2}-1$$

Ans. Let I = $\int_0^1 \sin^{-1} x\, dx$

Putting $x = \sin\theta$

$\Rightarrow$ $dx = \cos\theta\, d\theta$

Limits of integration when $x=0, \theta=0$ and when $x=1, \sin\theta=1$, i.e., $\theta=\frac{\pi}{2}$

$\therefore$ I = $\int_0^1 \sin^{-1} x\, dx$

= $\int_0^1 \theta\cos\theta\, d\theta$

= $\left[\theta\sin\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1.\sin\theta\, d\theta$

[Integrating by parts]

$\Rightarrow$ I = $\left(\frac{\pi}{2}-0\right)+\left[\cos\theta\right]_0^{\frac{\pi}{2}}$

= $\frac{\pi}{2}+\left(\cos\frac{\pi}{2}-\cos 0\right)$

$= \frac{\pi}{2} + (0-1) == \frac{\pi}{2} - 1$

Improper Integral

An **improper integral** is a definite **integral** that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration.

For the existence of Riemann integral (definite integral) R$_b$ f (x)dx, we require

that the limit of integration a and b are finite and function f (x) is bounded.
In case

(i) limit of integration a or b or both become infinite (improper integral of first kind),

(ii) integrand f (x) has singular points (discontinuity) i.e. f (x) becomes infinite at some points in the interval a ≤ x ≤ b (improper integral of second kind),

Problem 1: Evaluate the integral $\int_1^{\infty} \frac{dx}{x^5}$

Solution:

$$\begin{aligned}
\int_1^{\infty} \frac{dx}{x^5} &= \lim_{N\to\infty} \int_1^{N} \frac{dx}{x^5} \\
&= \lim_{N\to\infty} \left.\frac{-1}{4x^4}\right|_1^N \\
&= \lim_{N\to\infty} \frac{-1}{4N^4} - \frac{-1}{4(1)^4} \\
&= \frac{1}{4}
\end{aligned}$$

Problem2 : Evaluate the integral

$$\int_2^{\infty} \frac{dx}{(3+x)^3}$$

Solution:

$$\begin{aligned}
\int_2^{\infty} \frac{dx}{(3+x)^3} &= \lim_{N\to\infty} \int_2^{N} \frac{dx}{(3+x)^3} \\
&= \lim_{N\to\infty} \left.\frac{-1}{2(3+x)^2}\right|_2^N \\
&= \lim_{N\to\infty} \frac{-1}{2(3+N)^2} - \frac{-1}{2(3+\ 2\ )^2} \\
&= \frac{1}{50}
\end{aligned}$$

Problem3 : Evaluate the integral

$$\int_{-1}^{\infty} e^{-5x}\, dx$$

Solution:

$$\begin{aligned}
\int_{-1}^{\infty} e^{-5x}\, dx &= \lim_{N\to\infty} \int_{-1}^{N} e^{-5x}\, dx \\
&= \lim_{N\to\infty} \left.\frac{-1}{5} e^{-5x}\right|_{-1}^N \\
&= \lim_{N\to\infty} \frac{-1}{5} e^{-5N} - \frac{-1}{5} e^{-5(-1)} \\
&= \frac{e^5}{5}
\end{aligned}$$

Problem 4: Evaluate the integral

$$\int_0^{\infty} x e^{-3x^2}\, dx$$

Solution:

$$\int_0^{\infty} xe^{-3x^2}\,dx = \lim_{N\to\infty}\int_0^N xe^{-3x^2}\,dx$$

Substitution,

$$\begin{aligned} u &= -3x^2 \\ du &= -6x\,dx \end{aligned}$$

Hence,

$$\begin{aligned} \int xe^{-3x^2}\,dx &= \int \frac{-1}{6}e^u\,du \\ &= \frac{-1}{6}e^u \\ &= \frac{-1}{6}e^{-3x^2} \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{N\to\infty}\int_0^N xe^{-3x^2}\,dx &= \lim_{N\to\infty}\frac{-1}{6}e^{-3x^2}\Big|_0^N \\ &= \lim_{N\to\infty}\left(\frac{-e^{-3N^2}}{6}\right) - \frac{-e^{-3(0)^2}}{6} \\ &= \frac{1}{6} \end{aligned}$$

Problem 5: Evaluate the integral

$$\int_3^{\infty}\frac{dx}{x^2+9}$$

Solution:

$$\begin{aligned} \int_3^{\infty}\frac{dx}{x^2+9} &= \lim_{N\to\infty}\int_3^N\frac{dx}{x^2+9} \\ &= \lim_{N\to\infty}\frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_3^N \\ &= \lim_{N\to\infty}\frac{1}{3}\arctan\left(\frac{N}{3}\right) - \frac{1}{3}\arctan\left(\frac{3}{3}\right) \\ &= \frac{1}{3}\cdot\frac{\pi}{2} - \frac{\pi}{12} \\ &= \frac{\pi}{12} \end{aligned}$$

Problem 6: Evaluate the integral

$$\int_{-\infty}^{0} 2^{5x}\,dx$$

Solution:

$$\begin{aligned}\int_{-\infty}^{0} 2^{5x}\,dx &= \lim_{N\to\infty}\int_{-N}^{0} 2^{5x}\,dx \\ &= \lim_{N\to\infty}\frac{1}{5}\frac{2^{5x}}{\ln\ 2}\Bigg|_{-N}^{0} \\ &= \frac{2^{5\cdot 0}}{5\ln\ 2} - \lim_{N\to\infty}\frac{2^{5(-N)}}{5\ln\ 2} \\ &= \frac{1}{5\ln\ 2}\end{aligned}$$

Problem 7: Evaluate the integral

$$\int_{1}^{\infty}\frac{x\,dx}{(x^2+5)^3}$$

Solution:

$$\begin{aligned}\int_{1}^{\infty}\frac{x\,dx}{(x^2+5)^3} &= \lim_{N\to\infty}\int_{1}^{N}\frac{x\,dx}{(x^2+5)^3} \\ &= \lim_{N\to\infty}\frac{-1}{4(x^2+5)^2}\Bigg|_{1}^{N} \\ &= \lim_{N\to\infty}\frac{-1}{4(N^2+5)^2} - \frac{-1}{4(1^2+5)^2} \\ &= \frac{1}{144}\end{aligned}$$

Problem 8 : Evaluate the integral

$$\int_{-\infty}^{\infty}\frac{dx}{25+4x^2}$$

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dx}{25+4x^2} &= 2\int_{0}^{\infty}\frac{dx}{25+4x^2}\\ &= 2\lim_{N\to\infty}\int_{0}^{N}\frac{dx}{25+4x^2}\\ &= 2\lim_{N\to\infty}\left.\frac{\arctan\left(\frac{2x}{5}\right)}{10}\right|_{0}^{N}\\ &= 2\lim_{N\to\infty}\frac{\arctan\left(\frac{2N}{5}\right)}{10}-\frac{\arctan(0)}{10}\\ &= \frac{\pi}{10}\end{aligned}$$

Problem 9: Evaluate the integral

$$\int_{-\infty}^{\infty}\frac{dx}{25+4x^2}$$

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dx}{25+4x^2} &= 2\int_{0}^{\infty}\frac{dx}{25+4x^2}\\ &= 2\lim_{N\to\infty}\int_{0}^{N}\frac{dx}{25+4x^2}\\ &= 2\lim_{N\to\infty}\left.\frac{\arctan\left(\frac{2x}{5}\right)}{10}\right|_{0}^{N}\\ &= 2\lim_{N\to\infty}\frac{\arctan\left(\frac{2N}{5}\right)}{10}-\frac{\arctan(0)}{10}\\ &= \frac{\pi}{10}\end{aligned}$$

Gamma function

Gamma function is defined as $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$

Q 1 Prove $\Gamma(1) = 1$

Sol. $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$

$$\therefore \Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx = \int_0^\infty x^0e^{-x}dx = \int_0^\infty e^{-x}dx = \left|\frac{e^{-x}}{-1}\right|_0^\infty = e^0 - e^{-\infty} = 1 - 0$$
$$= 1$$

Beta function.

Beta function is defined as $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$dx

Q1 Write recurrence relation of Gamma function.

Sol. $\Gamma(n) = (n-1)\Gamma(n-1)$

Q2 Prove$\Gamma(n) = (n-1)!$

Proof: Using recurrence relation of Gamma function,

$$\Gamma(n) = (n-1)\Gamma(n-1) \xrightarrow{yields} \Gamma(n) = (n-1)[(n-2)\Gamma(n-2)]$$

$$\xrightarrow{\text{yields}} \Gamma(n) = (n-1)(n-2)[(n-3)\Gamma(n-3)]$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots ..$$

$$\xrightarrow{\text{yields}} \Gamma(n) = (n-1)(n-2)(n-3) \dots \dots \dots 1\Gamma(1)$$
$$= (n-1)(n-2)(n-3) \dots \dots \dots 1.1 = (n-1)!$$

Q3 Write relationship between Beta and Gamma functions.

Sol. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Q4 Prove $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}$

Sol. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ Put $x = \frac{1}{1+y} \xrightarrow{yields} dx = -\frac{dy}{(1+y)^2}$

$$\therefore \beta(m,n) = -\int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \left(1-\frac{1}{1+y}\right)^{n-1} \frac{dy}{(1+y)^2}$$

$$\xrightarrow{yields} \beta(m,n) = -\int_{\infty}^{0} \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(1+y)^{n-1}} \frac{dy}{(1+y)^2}$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Replacing y by x, we get $\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}$

Q4 Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\Gamma\frac{1}{4}}{4\Gamma\frac{3}{4}}$

Putting x^4= t the integral become

$$l = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \frac{\frac{1}{4}t^{\frac{-3}{4}}dt}{\sqrt{1-t}} = \frac{1}{4}\int_0^1 t^{\frac{1}{4}-1}(1-t)^{\frac{1}{2}-1}dt$$

$$= \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\Gamma\frac{1}{4}\Gamma\frac{1}{2}}{\Gamma\frac{1}{4}+\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{4}\frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

Applications of Definite Integral to Evaluate Surface Areas and Volume of Revolutions

1. **Length of curve in Cartesian form**
    - If the equation of curve is $y = f(x)$, length of arc of the curve is $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$
    - If the equation of curve is $x = g(y)$, length of arc of the curve is $\int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$
    -
2. **Length of curve in Polar form**

    If the equation of curve is r= $f(\theta)$, length of arc of the curve is $\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$
3. **Length of curve in Parametric form**

    If the equations of curve are $x = f(t)\ and\ y = g(t)$, length of arc of the curve is $\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

4. **Area under the curve**
    - Area bounded by the curve $y = f(x), x = a, x = b\ and\ x - axis$ is $\int_a^b |f(x)|\ dx$
    - Area bounded by the curve $x = g(y), y = a, y = b\ and\ y - axis$ is $\int_a^b |g(y)|\ dy$
    - Area bounded by the curve $y = f(x), y = g(x), x = a, x = b$ is $\int_a^b |f(x) - g(x)|\ dx$
    - If the curve is in parametric form, area is $\int_\alpha^\beta x \frac{dy}{dt}\ dt$

5. **Volume of revolution**
    - Volume obtained by revolution of curve $y = f(x)$bounded by $x = a, x = b$ about $x - axis$ is $\int_a^b \pi y^2 dx$
    - Volume obtained by revolution of curve $x = f(y)$bounded by $y = a, y = b$ about $y - axis$ is $\int_a^b \pi x^2 dy$
    - Volume obtained by revolution of curve $y = f(x)$bounded by $x = a, x = b$ about a line $y = p$ is $\int_a^b \pi (y - p)^2 dx$
    - Volume obtained by revolution of curve $x = f(y)$bounded by $y = a, y = b$ about a line $x = q$ is $\int_a^b \pi (x - q)^2 dy$
6. **Centre of gravity**
    - Centre of gravity of a uniform plane curve is

      $\bar{x} = \frac{\int_a^b x \frac{ds}{dx} dx}{\int_a^b \frac{ds}{dx} dx}and\bar{y} = \frac{\int_a^b y \frac{ds}{dx} dx}{\int_a^b \frac{ds}{dx} dx}$
    - Centre of gravity of a uniform plane area is

      $\bar{x} = \frac{\int_a^b xydx}{\int_a^b ydx}and\bar{y} = \frac{\int_a^b \frac{1}{2} y^2 dx}{\int_a^b ydx}$

- Centre of gravity of a volume of the solid of revolution about $x - axis$ is

  $\bar{x} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx}$ and $\bar{y} = 0$

- Centre of gravity of a surface area of the solid of revolution about $x - axis$ is $\bar{x} = \frac{\int_a^b xy\frac{ds}{dx}dx}{\int_a^b y\frac{ds}{dx}dx}$ and $\bar{y} = 0$

- Centre of gravity of a sectorial area subtending an angle θ rt the centre is $\bar{x} = \frac{\int_\alpha^\beta \frac{2}{3}r^3 \cos\theta d\theta}{\int_\alpha^\beta r^2 d\theta}$ and $\bar{y} = \frac{\int_\alpha^\beta \frac{2}{3}r^3 \sin\theta d\theta}{\int_\alpha^\beta r^2 d\theta}$

1. Rectify the curve $x = a\, cos^3 t\,,\quad y = a\, sin^3 t$

Sol. $= a\, cos^3 t\,,\quad y = a\, sin^3 t$

$\frac{dx}{dt} = -3\,\mathrm{a} \sin t\, cos^2 t\;,\quad \frac{dy}{dt} = 3\,\mathrm{a}\cos t\, sin^2 t$

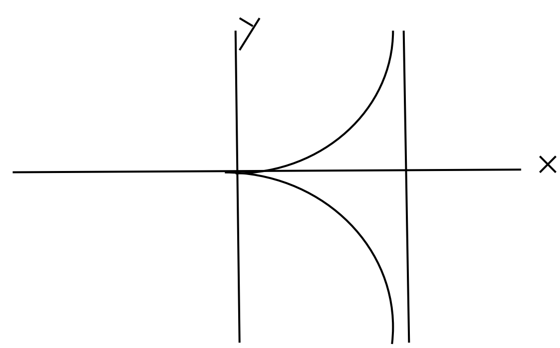
$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2 cos^2 t sin^2 t\,(sin^2 t + cos^2 t) = 9a^2 cos^2 t\, sin^2 t$

Length of whole curve is $= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 12a\int_0^{\frac{\pi}{2}} \cos t \sin t\, dt$

$= 6a\int_0^{\frac{\pi}{2}} \sin 2t\, dt = 6\mathrm{a}$

2.Find the area between the curve $y^2(2a - x) = x^3$ and its asymptote.

Sol. The given curve is symmetrical about x-axis and passes through (0, 0). Asymptote parallel to y-axis is given by Coefficient $(y^2) = 0$ i.e. 2a-x = 0 i.e. x=2a is the asymptote parallel to y-axis.



Required area $= 2\int_0^{2a} y dx = 2\int_0^{2a} \frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}$

Put x = 2a $sin^2\theta \xrightarrow{yields} dx = 4asin\theta \cos\theta\, d\theta$

x = 0 $\xrightarrow{yields} \theta = 0$ and x = 2a $\xrightarrow{yields} \theta = \frac{\pi}{2}$

Required area $= 2\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{3}{2}} sin^3\theta(4asin\theta \cos\theta \, d\theta)}{\sqrt{2a}cos\theta} = 16a^2 \int_0^{\frac{\pi}{2}} sin^4\theta d\theta = 16a^2 \left(\frac{3*1}{4*2}\right)\frac{\pi}{2}$

$= 3\pi a^2$

3. Find the volume of the solid generated by revolution of the curve r = a (1+ cos θ) about initial line.

Sol. Required volume = $\pi \int_0^{2a} y^2 dx$

Change x,y to polar coordinates.

Put x = r cos θ and y = r sin θ

Therefore x = a (1+ cos θ) cos θ and y = a (1+ cos θ) sin θ

dx = a [- sin θ cos θ - sin θ - sin θ cos θ] d θ = a[- 2sin θ cos θ - sin θ] d θ

x = 0 $\xrightarrow{yields} \theta = 0$ and x= 2a$\xrightarrow{yields} \theta = \pi$

Volume = $\pi \int_0^{\pi} a^2(1+ \cos\theta)^2 sin^2\theta \, a(-\sin\theta - 2\sin\theta\cos\theta) \mathrm{d}\,\theta$

$$= -\pi a^3 \int_0^{\pi} (1+ \cos\theta)^2 sin^3\theta \, (1 + 2\cos\theta) \mathrm{d}\,\theta$$

$$= -\pi a^3 \int_0^{\pi} 4cos^4\frac{\theta}{2} 8sin^3\frac{\theta}{2} cos^3\frac{\theta}{2}\left(1 + 2\left(2\cos^2\frac{\theta}{2} - 1\right)\right) \mathrm{d}\,\theta$$

Put $\frac{\theta}{2} = \emptyset$, Therefored $\theta = 2d\emptyset$

Hence Volume = $= -\pi a^3 \int_0^{\frac{\pi}{2}} 32\, cos^7\emptyset sin^3\emptyset (4\cos^2\emptyset - 1) 2d\emptyset$

$$= -2\pi a^3 \left[128 \int_0^{\frac{\pi}{2}} cos^9\emptyset sin^3\emptyset \, d\emptyset - 32 \int_0^{\frac{\pi}{2}} cos^7\emptyset sin^3\emptyset \, d\emptyset\right]$$

$== -2\pi a^3 \left[128\left(\frac{8.6.4.2.2}{12.10.8.6.4.2}\right) - 32\left(\frac{6.4.2.2}{10.8.6.4.2}\right)\right] = \frac{8\pi a^3}{3}$

4.Find moment of inertia of one loop of lemniscates$r^2 = a^2 \cos 2\theta$ about initial line.

Sol. Let M be the mass and ρ, the density of one loop of the leminscate $r^2 = a^2 \cos 2\theta$.

For the loop, θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ and r varies from 0 to $a^2\sqrt{\cos 2\theta}$

$$M = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} \rho\, r\, dr\, d\theta \;=\; \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \rho \left|\frac{r^2}{2}\right|_0^{a\sqrt{\cos 2\theta}} d\theta \;=\frac{\rho}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta$$

$$=\frac{\rho a^2}{2}.\,2.\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta \;=\frac{\rho a^2}{2}$$

M.I. about initial line is given by,

$$I_x = \iint \rho\, y^2\, dx\, dy \;=\; \iint \rho\, r^2 \sin^2\theta.\, r.\, dr\, d\theta$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} \rho \sin^2\theta r^3\, dr\, d\theta \;=\; \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \rho \sin^2\theta \left[\frac{r^4}{4}\right] d\theta$$

$$= \rho\,\frac{a^4}{4}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta \cos^2 2\theta\, d\theta = \frac{\rho\, a^4}{4}\, 2\int_0^{\frac{\pi}{4}} \sin^2\theta \cos^2 2\theta\, d\theta$$

$$= \rho\,\frac{a^2}{2}\int_0^{\frac{\pi}{2}} \sin^2\frac{\phi}{2} \cos^2\frac{\phi}{2}\frac{1}{2}\, d\phi, \text{ where } \theta = \frac{\phi}{2}$$

$$= \frac{\rho\, a^4}{2}\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos\phi}{2}\right)\cos^2\phi\, d\phi \;=\; \frac{\rho\, a^4}{4}\int_0^{\frac{\pi}{2}}(\cos^2\phi - \cos^3\phi)\, d\phi$$

$$= \frac{\rho\, a^4}{8}\left[\frac{1}{2}.\frac{\pi}{2} - \frac{2}{3}\right] \;=\; \frac{\rho\, a^4}{96}(3\pi - 8)$$

$$I = \frac{M\, a^2}{48}(3\pi - 8)$$

5.Find length of any arc of the curve $r = a\sin^2\frac{\theta}{2}$

Sol. The given curve is $r = a\,(\sin\frac{\theta}{2})^2$

Now $\frac{dr}{d\theta} = 2a\,\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{1}{2} = a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

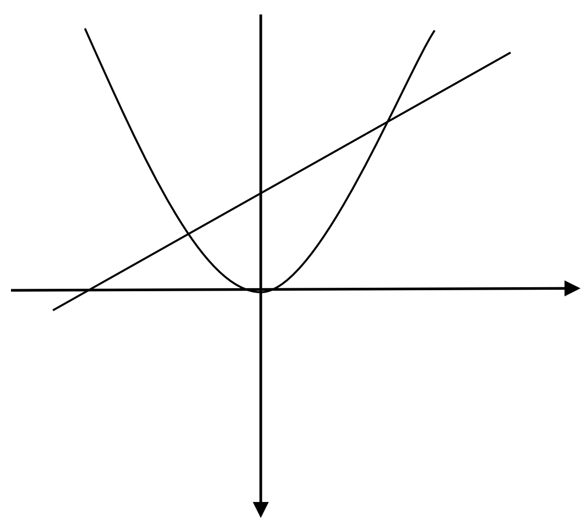
Thus, Length of any arc = $\int_{\theta 1}^{\theta 2}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta = \int_{\theta 1}^{\theta 2}\sqrt{a^2\,(\sin\frac{\theta}{2})^4 + a^2(\cos\frac{\theta}{2})^2\,(\sin\frac{\theta}{2})^2}\, d\theta$

$$= a\int_{\theta 1}^{\theta 2}\sqrt{(\sin\frac{\theta}{2})^2 + \left(\cos\frac{\theta}{2}\right)^2}\,.\sin\frac{\theta}{2}\, d\theta = a\int_{\theta 1}^{\theta 2}\sin\frac{\theta}{2}\, d\theta$$

$$= a\left[-\frac{\cos\frac{\theta}{2}}{\frac{1}{2}}\right]_{\theta 1}^{\theta 2} = -2a\left[\cos\frac{\theta 2}{2} - \cos\frac{\theta 1}{2}\right] = 2a\left(\cos\frac{\theta 1}{2} - \cos\frac{\theta 2}{2}\right)$$

6.Findcentre of gravity of a plate whose density $\rho(x, y)$ is constant and is bounded by the curves $y = x^2$ and $y = x + 2$. Also find moment of inertia about x-axis.

Sol. $y = x^2$ is an upward parabola with vertex as origin and intersects the line $y = x + 2$ at the points (-1,1) and (2,4).



Given the density ρ is constant then the C.G. of the area is $\bar{x} = \frac{\int_{-1}^{2}\int_{x^2}^{x+2} x\rho\, dy\, dx}{\int_{-1}^{2}\int_{x^2}^{x+2} \rho\, dy\, dx}$

$$= \frac{\int_{-1}^{2} x\,(y)_{x^2}^{x+2}\, dx}{\int_{-1}^{2}(y)_{x^2}^{x+2}\, dx} = \frac{\int_{-1}^{2} x\,(x + 2 - x^2)\, dx}{\int_{-1}^{2}\,(x + 2 - x^2)\, dx} = \frac{\int_{-1}^{2}(x^2 + 2x - x^3)\, dx}{(\frac{x^2}{2} + 2x - \frac{x^3}{3})_{-1}^{2}}$$

$$= \frac{(\frac{x^3}{3} + x^2 - \frac{x^4}{4})_{-1}^{2}}{\frac{9}{2}} = \frac{\frac{9}{4}}{\frac{9}{2}} = \frac{1}{2}$$

Now for moment of inertia, Let M be the mass of the arc then

$$M = \int_{-1}^{2}\int_{x^2}^{x+2} \rho\, dy\, dx = \frac{9\rho}{2}$$

Or $\rho = \frac{2M}{9}$

So, M.I. about x-axis of the arc $= \int_{-1}^{2}\int_{x^2}^{x+2} \rho y^2\, dy\, dx = \rho\int_{-1}^{2}(\frac{y^3}{3})_{x^2}^{x+2}\, dx$

$$= \frac{\rho}{3}\int_{-1}^{2}((x+2)^3 - (x^2)^{\,3})\; dx$$

$$= \frac{2M}{27}\int_{-1}^{2}(x^3 + 8 + 6x^2 + 12x - x^6)\; dx$$

$$= \frac{2M}{27}(\frac{x^4}{4} + 8x + 2x^3 + 6x^2 - \frac{x^7}{7})_{-1}^{2} = \frac{2M}{27}.\frac{1273}{28} = \frac{1273M}{378}$$

7.Find the moment of inertia of one loop of the leminscate $r^2 = a^2 \cos 2\theta$.about the initial line.

Sol. Let M be the mass and ρ, the density of one loop of the leminscate $r^2 = a^2 \cos 2\theta$.

For the loop, θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ and r varies from 0 to $a^2\sqrt{\cos 2\theta}$

$M = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{a\sqrt{\cos 2\theta}} \rho\, r\, dr\, d\theta = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \rho \left|\frac{r^2}{2}\right|_{0}^{a\sqrt{\cos 2\theta}} d\theta = \frac{\rho}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta$

$= \frac{\rho a^2}{2} . 2. \int_{0}^{\frac{\pi}{4}} \cos 2\theta\, d\theta = \frac{\rho a^2}{2}$

M.I. about initial line is given by,

$I_x = \iint \rho\, y^2\, dx\, dy = \iint \rho\, r^2 \sin^2\theta . r. dr\, d\theta$

$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{a\sqrt{\cos 2\theta}} \rho \sin^2\theta r^3\, dr\, d\theta = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \rho \sin^2\theta \left[\frac{r^4}{4}\right] d\theta$

$= \rho\,\frac{a^4}{4}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta \cos^2 2\theta\, d\theta = \frac{\rho\, a^4}{4}\; 2\int_{0}^{\frac{\pi}{4}} \sin^2\theta \cos^2 2\theta\, d\theta$

$= \rho\,\frac{a^2}{2}\int_{0}^{\frac{\pi}{2}} \sin^2\frac{\emptyset}{2} \cos^2\frac{\emptyset}{2}\frac{1}{2} d\phi$, where $\theta = \frac{\emptyset}{2}$

$= \frac{\rho\, a^4}{2}\int_{0}^{\frac{\pi}{2}}\left(\frac{1-\cos\emptyset}{2}\right)\cos^2\phi\, d\phi = \frac{\rho\, a^4}{4}\int_{0}^{\frac{\pi}{2}}(\cos^2\emptyset - \cos^3\emptyset)\, d\phi$

$= \frac{\rho\, a^4}{8}\left[\frac{1}{2} . \frac{\pi}{2} - \frac{2}{3}\right] = \frac{\rho\, a^4}{96}(3\pi - 8)$

$$I = \frac{M\,a^2}{48}(\,3\pi - 8)$$