Exercise 2.12

1. Prove that $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=0.$
2. Prove that
$$\lim_{x\to a}\left[\frac{1}{x-a}-\cot(x-a)\right]=0.$$
3. Prove that
$$\lim_{x\to \frac{\pi}{2}}(\sec x-\tan x)=0.$$
4. Prove that
$$\lim_{x\to \frac{\pi}{2}}\left(\tan x-\frac{2x\sec x}{\pi}\right)=\frac{2}{\pi}.$$
5. Prove that
$$\lim_{x\to a}\left[\frac{1}{x-a}-\frac{1}{\log(x+1-a)}\right]=-\frac{1}{2}.$$
6. Prove that
$$\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{1}{\log(x-2)}\right]=-\frac{1}{2}.$$
7. Prove that $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\frac{1}{2}.$
8. Prove that $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\frac{1}{2}.$

2.10.7 Type 5 : $1^\infty, \infty^0, 0^0$

To evaluate the limits of the type $\lim_{x\to a}[f(x)]^{g(x)}$ which takes any one of the above form, we proceed as follows:

Let
$$l=\lim_{x\to a}[f(x)]^{g(x)}$$
$$\log l=\lim_{x\to a}[g(x)\cdot\log f(x)] \quad [\text{if } f(x)>0]$$

which takes the form $\infty\times 0$, i.e., type 3 form.

Example 1: Prove that $\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}=ae.$

Solution: Let $l=\lim_{x\to 0}(a^x+x)^{\frac{1}{x}}$ $\qquad [1^\infty]$

$$\log l=\lim_{x\to 0}\frac{1}{x}\cdot\log(a^x+x)$$
$$=\lim_{x\to 0}\frac{\log(a^x+x)}{x} \qquad \left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{\frac{1}{a^x+x}(a^x\log a+1)}{1} \qquad \text{[Applying L'Hospital's rule]}$$
$$=\frac{a^0\log a+1}{a^0+0}=\frac{\log_e a+\log_e e}{1}$$

Hence,
$$\log l=\log ae$$
$$l=ae$$

Example 2: Prove that $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}=\sqrt{ab}.$

Solution: Let $l=\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$ $\qquad [1^\infty]$

$$\log l=\lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x+b^x}{2}\right)$$
$$=\lim_{x\to 0}\frac{\log\left(\frac{a^x+b^x}{2}\right)}{x} \qquad \left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\left(\frac{2}{a^x+b^x}\right)\cdot\frac{(a^x\log a+b^x\log b)}{2} \qquad \text{[Applying L'Hospital's rule]}$$
$$=\left(\frac{2}{a^0+b^0}\right)\frac{(a^0\log a+b^0\log b)}{2}$$
$$=\frac{1}{2}\cdot\log ab$$

Hence,
$$\log l=\log(ab)^{\frac{1}{2}}$$
$$l=\sqrt{ab}$$

Example 3: Prove that $\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}+d^{\frac{1}{x}}}{4}\right)^x=(abcd)^{\frac{1}{4}}.$

Solution: Let $l=\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}+d^{\frac{1}{x}}}{4}\right)^x$

Taking $\frac{1}{x}=y$, when $x\to\infty, y\to 0$

$$l=\lim_{y\to 0}\left(\frac{a^y+b^y+c^y+d^y}{4}\right)^{\frac{1}{y}} \qquad [1^\infty]$$

$$\log l = \lim_{y\to 0}\frac{1}{y}\log\left(\frac{a^y+b^y+c^y+d^y}{4}\right)$$

$$= \lim_{y\to 0}\frac{\log\left(\frac{a^y+b^y+c^y+d^y}{4}\right)}{y} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0}\left(\frac{4}{a^y+b^y+c^y+d^y}\right)\left(\frac{a^y\log a+b^y\log b+c^y\log c+d^y\log d}{4}\right)$$

[Applying L'Hospital's rule]

$$= \frac{\log a+\log b+\log c+\log d}{4}$$

$$= \frac{1}{4}\log(abcd)$$

Hence, $\log l = \log(abcd)^{\frac{1}{4}}$

$$l = (abcd)^{\frac{1}{4}}$$

Example 4: Prove that $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x = e^{\frac{2}{a}}$.

Solution: Let $l = \lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x$

$$= \lim_{x\to\infty}\left(\frac{1+\frac{1}{ax}}{1-\frac{1}{ax}}\right)^x \qquad [1^\infty]$$

$$\log l = \lim_{x\to\infty} x\log\left(\frac{1+\frac{1}{ax}}{1-\frac{1}{ax}}\right)$$

$$= \lim_{x\to\infty}\frac{ax}{a}\left[\log\left(1+\frac{1}{ax}\right)-\log\left(1-\frac{1}{ax}\right)\right]$$

$$= \lim_{x\to\infty}\frac{1}{a}\left[\log\left(1+\frac{1}{ax}\right)^{ax}+\log\left(1-\frac{1}{ax}\right)^{-ax}\right]$$

$$= \frac{1}{a}(\log e+\log e) = \frac{1}{a}(1+1) = \frac{2}{a} \qquad \left[\because \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax} = e\right]$$

Hence, $\log l = \frac{2}{a}$

$$l = e^{\frac{2}{a}}$$

Example 5: Prove that $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}} = e^{\frac{2}{\pi}}$.

Solution: Let $l = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ $\qquad [1^\infty]$

$$\log l = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)}{\left(-\operatorname{cosec}^2\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \frac{2}{\pi}$$

Hence, $\log l = \frac{2}{\pi}$

$$l = e^{\frac{2}{\pi}}$$

Example 6: Prove that $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$.

Solution: Let $l = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ $\qquad [1^\infty] \quad \left[\because \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x}{\tan x}\left(\frac{x\sec^2 x-\tan x}{x^2}\right)\cdot\frac{1}{2x} \qquad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{2x^3} \quad \left[\frac{0}{0}\right] \quad \left[\because \lim_{x\to 0} \frac{x}{\tan x} = 1\right]$$

$$= \lim_{x\to 0} \frac{\sec^2 x + x\cdot 2\sec^2 x \tan x - \sec^2 x}{6x^2} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0} \frac{\sec^2 x}{3}\cdot\frac{\tan x}{x} = \frac{1}{3}$$

Hence, $\log l = \frac{1}{3}$

$$l = e^{\frac{1}{3}}.$$

Example 7: Prove that $\lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{6}}$.

Solution: Let $l = \lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}}$ $[1^\infty]$ $\left[\because \lim_{x\to 0}\frac{\sinh x}{x} = 1\right]$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\sinh x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\sinh x}{x}\right)}{x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{x}{\sinh x}\left(\frac{x\cosh x - \sinh x}{x^2}\right)\cdot\frac{1}{2x} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{x\cosh x - \sinh x}{2x^3} \quad \left[\frac{0}{0}\right] \quad \left[\because \lim_{x\to 0}\frac{x}{\sinh x} = 1\right]$$

$$= \lim_{x\to 0}\frac{x\sinh x + \cosh x - \cosh x}{6x^2} \quad \text{[Applying L'Hospital's rule]}$$

$$= \lim_{x\to 0}\frac{1}{6}\cdot\frac{\sinh x}{x} = \frac{1}{6} \quad \left[\because \lim_{x\to 0}\frac{\sinh x}{x} = 1\right]$$

Hence, $\log l = \frac{1}{6}$

$$l = e^{\frac{1}{6}}$$

Example 8: Prove that $\lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x} = 1$.

Solution: Let $l = \lim_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x}$ $[\infty^0]$

$$\log l = \lim_{x\to 0}(1-\cos x)\log\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\left(2\sin^2\frac{x}{2}\right)(-\log x)$$

$$= \lim_{x\to 0}\frac{2\left(\sin\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2}(-\log x)$$

$$= \lim_{x\to 0}\frac{x^2(-\log x)}{2} \quad \left[\because \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) = 1\right]$$

$$= \lim_{x\to 0}\frac{1}{2}\frac{(-\log x)}{\left(\frac{1}{x^2}\right)} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\frac{-1}{x}}{\frac{-2}{x^3}}\right) \quad \text{[Applying L'Hospital's rule]}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{x^2}{2}\right) = 0$$

Hence, $\log l = 0$

$$l = e^0 = 1$$

Example 9: Prove that $\lim_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}} = e$.

Solution: Let $l = \lim_{x\to\infty} e^{\frac{\sinh^{-1}x}{\cosh^{-1}x}}$

$$= \lim_{x\to\infty}\left(e^{\sinh^{-1}x}\right)^{\frac{1}{\cosh^{-1}x}} \quad [\infty^0]$$

$$\log l = \lim_{x\to\infty}\frac{\sinh^{-1}x}{\cosh^{-1}x}\cdot\log e$$

$$= \lim_{x\to\infty}\frac{\log(x+\sqrt{x^2+1})}{\log(x+\sqrt{x^2-1})} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\sqrt{x^2+1}}\cdot 2x\right)}{\frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right)}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}} = 1$$

Hence, $\log l = 1$

$l = e^1 = e.$

Example 10: Prove that $\lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{x}} = 1$.

Solution: Let $l = \lim_{x\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{x}}$ $[0^0]$

$$\log l = \lim_{x\to\infty} \frac{1}{x}\log\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{-\log x}{x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= -\lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$ [Applying L'Hospital's rule]

$= 0$

Hence, $\log l = 0$

$l = e^0 = 1$

Example 11: Prove that $\lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}} = e$.

Solution: Let $l = \lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $[0^0]$

$$\log l = \lim_{x\to 1} \frac{1}{\log(1-x)} \log(1-x^2) \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1} \frac{\frac{-2x}{(1-x^2)}}{\frac{1}{(1-x)}(-1)}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to 1} \frac{2x(1-x)}{(1-x)(1+x)} = \lim_{x\to 1} \frac{2x}{1+x} = 1$$

Hence, $\log l = 1$

$l = e$

Example 12: Prove that $\lim_{x\to\infty} \frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = 1$.

Solution: Let $l = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^x$

$$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2} \qquad (\infty^0)$$

$$\log l = \lim_{x\to\infty} x^2 \log\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{2}{x^3}}$$ [Applying L'Hospital's rule]

$$= \lim_{x\to\infty} \frac{x}{2\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x^2}{2(x+1)} = 0$$

Hence, $\log l = 0$

$l = e^0 = 1$

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^x = 1$$

Hence, $$\lim_{x\to\infty} \frac{e^x}{\left[\left(1+\frac{1}{x}\right)^x\right]^x} = \frac{e^0}{1} = \frac{1}{1} = 1.$$

Example 13: Prove that $\lim_{x\to 0}\dfrac{1-x^{\sin x}}{x\log x}=-1$.

Solution: Let $l_1=\lim_{x\to 0}x^{\sin x}$ $\quad[0^0]$

$$\log l_1=\lim_{x\to 0}\sin x\cdot\log x=\lim_{x\to 0}\frac{\log x}{\operatorname{cosec} x}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{\frac{1}{x}}{-\operatorname{cosec} x\cot x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}-\frac{\sin^2 x}{x\cos x}=-\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{x}{\cos x}=0\qquad\left[\because\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$\log l_1=0,\ l_1=e^0=1,\ \therefore\ \lim_{x\to 0}x^{\sin x}=1\qquad\ldots(1)$$

Let $l_2=\lim_{x\to 0}x\log x$ $\quad[0\times\infty]$

$$=\lim_{x\to 0}\frac{\log x}{\frac{1}{x}}\qquad\left[\frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}(-x)=0\qquad\ldots(2)$$

Let $l=\lim_{x\to 0}\dfrac{1-x^{\sin x}}{x\log x}$ $\quad\left[\dfrac{0}{0}\right]$ $\quad$ [Using Eqs (1) and (2)]

$$=\lim_{x\to 0}\frac{1-e^{\sin x\log x}}{x\log x}$$

$$\lim_{x\to 0}\frac{1-x^{\sin x}}{x\log x}=\lim_{x\to 0}\frac{-e^{\sin x\log x}\left(\frac{\sin x}{x}+\cos x\cdot\log x\right)}{1+\log x}\qquad\text{[Applying L'Hospital's rule]}$$

$$=\lim_{x\to 0}\frac{-x^{\sin x}\left[\left(\frac{\sin x}{x}\right)\cdot\frac{1}{\log x}+\cos x\right]}{\frac{1}{\log x}+1}\qquad\ldots(3)$$

[Dividing numerator and denominator by $\log x$]

$$=-\frac{1\left(1\cdot\frac{1}{\infty}+\cos 0\right)}{\frac{1}{\infty}+1}=-1\qquad\left[\text{Using Eq. (1) and }\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Exercise 2.13

1. Prove that
$$\lim_{x\to 0}\left(\frac{a_1^x+a_2^x+\ldots+a_n^x}{n}\right)^{\frac{1}{x}}=(a_1a_2\ldots a_n)^{\frac{1}{n}}.$$

2. Prove that
$$\lim_{x\to\infty}\left(\frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{4}\right)^{4x}=24.$$

3. Prove that $\lim_{x\to 0}(1+\sin x)^{\frac{1}{2x}}=e^{\frac{1}{2}}$.

4. Prove that $\lim_{x\to 1}(x)^{\frac{1}{1-x}}=\frac{1}{e}$.

5. Prove that $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=1$.

6. Prove that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}=e^{-\frac{1}{6}}$.

7. Prove that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=e^a$.

8. Prove that $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x=e^2$.

9. Prove that $\lim_{x\to\infty}\left(\frac{2x+1}{2x-1}\right)^x=e$.

10. Prove that $\lim_{x\to 0}(1+\sin x)^{\operatorname{cosec} x}=e$.

11. Prove that $\lim_{x\to 0}(1+\sin x)^{\cot x}=e$.

12. Prove that $\lim_{x\to 0}(1+\tan x)^{\cot x}=e$.

13. Prove that $\lim_{x\to 0}(1-\tan x)^{\frac{1}{x}}=\frac{1}{e}$.

14. Prove that $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}=\frac{1}{\sqrt{e}}$.

15. Prove that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}=0$.

16. Prove that $\lim_{x\to 0}(\cos ax)^{\operatorname{cosec}^2 bx}=e^{-\frac{a^2}{2b^2}}$.

17. Prove that $\lim_{x\to 0}(\cos x)^{\cot^2 x}=\frac{1}{\sqrt{e}}$.

18. Prove that $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}=1$.

19. Prove that $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=e^2$.

20. Prove that $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\cot(x-a)}=e^{-\frac{1}{a}}$.

21. Prove that $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\cos 2x}=1$.

22. Prove that $\lim_{x\to\frac{\pi}{2}}(\sec x)^{\cot x}=1$.

23. Prove that $\lim_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x}=1$.

24. Prove that $\lim_{x\to 0}(\sin x)^{\tan x}=1$.

25. Prove that $\lim_{x\to 0}(1-x^n)^{\frac{1}{\log(1-x)}}=e$.

26. Prove that
$$\lim_{x\to a}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}=1.$$

27. Prove that $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)^{x^2}=e^{-\frac{1}{6}}$.

28. Prove that $\lim_{x\to 1}x^{\tan\left(\frac{\pi x}{2}\right)}=e$.

29. Prove that
$$\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}=e^{-\frac{a^2}{b^2}}.$$

30. Prove that $\lim_{x\to 0}(e^{3x}-5x)^{\frac{1}{x}}=e^{-2}$.

31. Prove that $\lim_{x\to 0}(\cos 2x)^{\left(\frac{1}{x^2}\right)}=e^{-2}$.

32. Prove that $\lim_{x\to 0}(\cot x)^{\sin x}=1$.