Matrices

Matrix:A matrix is a rectangular array of numbers, algebraic symbols, or mathematical functions, provided that such arrays are added and multiplied according to certain rules.

Order or size of Matrix: Order or size of Matrix $= \text{no. of rows} \times \text{no. of columns} = m \times n$

Where m is number of rows and n is number of columns.

Row Matrix:A matrix with one row is called a row matrix.

Column Matrix: A matrix with one column is called a row matrix.

Square matrix: A matrix is called a square matrix if the number of its rows equals the number of columns.

Rectangular Matrix:A matrix is called a rectangular matrix if the number of its rows is not equal to the number of columns.

Diagonal Matrix:A matrix is called a diagonal matrix if all its off-diagonal elements are equal to zero, but at least one of the diagonal elements is nonzero:

$$a_{ij} = 0 \text{ if } i \neq j$$

Identity Matrix: an identity matrix is a diagonal matrix whose diagonal elements are equal to unity.

Zero Matrix: A matrix is called a zero-matrix (or 0-matrix) if all its elements are equal to zero.

Transpose of Matrix: A matrix obtained from matrix A by interchanging its rows and columns is called transpose of A and is denoted by A^T or A'.

e.g. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

Upper Triangular Matrix: A square matrix A is called upper triangular matrix if all the elements below the principal diagonal are zero.

e.g. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

Lower Triangular Matrix: A square matrix A is called lower triangular matrix if all the elements above the principal diagonal are zero.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 2 & -3 & 6 \end{bmatrix}$

Symmetric matrix: A square matrix is called symmetric matrix if $A = A^T$

i.e.$a_{ij} = a_{ji}$

e.g.$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

SkewSymmetric matrix: A square matrix is called symmetric matrix if $A = -A^T$

i.e. $a_{ij} = -a_{ji}$. The diagonal elements of a skew-symmetric matrix are zero because $a_{ii} = -a_{ii}$ if and only if $a_{ii} = 0$

e.g. $\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

Matrix Addition: Only matrix of same dimension can be added. In case of matrix addition, the corresponding elements of two similar matrices are added to each other.

e.g. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+(-7) & 1+1 \\ 1+6 & 0+(-8) & 3+(-1) \\ 2+3 & -3+2 & 0+(-6) \end{bmatrix} = \begin{bmatrix} 6 & -5 & 2 \\ 7 & -8 & 2 \\ 5 & -1 & -6 \end{bmatrix}$

Matrix addition is both commutative and associative.

Scalar multiplication: The product of a scalar “k” and a matrix$A_{m\times n}$ is the matrix $kA_{m\times n}$ each of whose entries are “k” times the corresponding entry in $A_{m\times n}$.

e.g. $5\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 5\times 1 & 5\times 2 & 5\times 1 \\ 5\times 1 & 5\times 0 & 5\times 3 \\ 5\times 2 & 5\times(-3) & 5\times 0 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 5 \\ 5 & 0 & 15 \\ 10 & -15 & 0 \end{bmatrix}$

Matrix Multiplication: Two matrices A and B can be multiplied i.e. AB is possible if number of columns in A is equal to number of rows in B.

If order of A is $m \times n$ and order of B is $n \times p$, then order of AB is $m \times p$.

Matrix multiplication is not commutative i.e. $AB \neq BA$.

Determinant: The association of a real number with square matrices of any dimension (order) is called the determinant of the matrix. Determinants can be distinguished from a matrix because they are always enclosed within a single set of vertical lines $' \,||\, '$.

Properties of Determinants:

- Let A be a square matrix of order n. The sum of product of elements of any row (or column) with their cofactors is always equal to $|A|$ i.e. $\sum_{j=1}^{n} a_{ij}C_{ij} = |A|$
- Let A be a square matrix of order n. The sum of product of elements of any row (or column) with the cofactors of the corresponding elements of some other row (or column) is always equal to 0 i.e. $\sum_{j=1}^{n} a_{ij}C_{kj} = 0$
- $|A| = |A^T|$
- By interchanging any two rows (or columns), the value of determinant changes by minus sign.
- If any two rows (or columns) are identical, then $|A| = 0$.
- If each element of a row (or column) is multiplied by a constant k, then value of new determinant is k times the value of original determinant.
- If each element of a row (or column) is multiplied by a constant k and then added to the corresponding elements of some other row (or column), ten value of the determinant remains unchanged.
- If each element of a row (or column) is expressed as a sum of two or more terms, then the determinant can be expressed as sum of two or more determinants.
- If each element in a row (or column) is zero, then $|A| = 0$.
- If A and B are two square matrices, then $|AB| = |A||B|$.

Q. Evaluate $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ without expanding.

Sol. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$)

$\xrightarrow{\text{yields}} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$ (Taking out$(b-a)$ and $(c-a)$ common from R_2 and R_3)

$$\xrightarrow{\text{yields}} (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} (\text{Operating } R_3 \to R_3 - R_2)$$

$$\xrightarrow{\text{yields}} (b-a)(c-a)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 0 & c-b\end{vmatrix} \text{(Taking out } (c-b) \text{ common from } R_3)$$

$$\xrightarrow{\text{yields}} (b-a)(c-a)(c-b)\begin{vmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 0 & 1\end{vmatrix} \text{(Expanding along } C_1)$$

$$= (a-b)(b-c)(c-a)$$

Q. Evaluate $\begin{vmatrix}1 & a & a^2-bc\\ 1 & b & b^2-ac\\ 1 & c & c^2-ab\end{vmatrix}$ without expanding.

Sol. $\begin{vmatrix}1 & a & a^2-bc\\ 1 & b & b^2-ac\\ 1 & c & c^2-ab\end{vmatrix} = \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} + \begin{vmatrix}1 & a & -bc\\ 1 & b & -ac\\ 1 & c & -ab\end{vmatrix}$

(Taking out (-1) common from C_3 of second determinant)

$$= \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} - \begin{vmatrix}1 & a & bc\\ 1 & b & ac\\ 1 & c & ab\end{vmatrix}$$

(multiplying R_1, R_2, R_3 of second determinant by a, b, c respectively)

$= \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} - \frac{1}{abc}\begin{vmatrix}a & a^2 & abc\\ b & b^2 & abc\\ c & c^2 & abc\end{vmatrix}$ (Taking out (abc) common from C_3 of second determinant)

$$= \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} - \frac{abc}{abc}\begin{vmatrix}a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix} = \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} - \begin{vmatrix}a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix}$$

(Interchanging C_2 and C_3 in second determinant)

$$= \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} - \begin{vmatrix}a & 1 & a^2\\ b & 1 & b^2\\ c & 1 & c^2\end{vmatrix}$$

(Interchanging C_1 and C_2 in second determinant)

$$= \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} - \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} = 0$$

$\xrightarrow{\text{yields}} (b-a)(c-a)\begin{vmatrix}1 & a & a^2\\0 & 1 & b+a\\0 & 0 & c-b\end{vmatrix}$(Taking out $(c-b)$ common from R_3)

$\xrightarrow{\text{yields}} (b-a)(c-a)(c-b)\begin{vmatrix}1 & a & a^2\\0 & 1 & b+a\\0 & 0 & 1\end{vmatrix}$(Expanding along C_1)

$$= (a-b)(b-c)(c-a)$$

Inverse of Matrix

Let A be a square matrix. Then $A^{-1} = \frac{1}{|A|}\text{adj}.A.$

Note: $AA^{-1} = I = A^{-1}A$

Q1.Calculate A^{-1} if $A = \begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$

Here $A = \begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$

$|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1 \neq 0$. Thus A^{-1} exists.

Cofactors of elements a_{ij} in A are

$$c_{11} = (-1)^{1+1}\begin{vmatrix}4 & 3\\3 & 4\end{vmatrix} = 7$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} = -1$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix}1 & 4\\1 & 3\end{vmatrix} = -1$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix}3 & 3\\3 & 4\end{vmatrix} = -3$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} = 1$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix}1 & 3\\1 & 3\end{vmatrix} = 0$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix}3 & 3\\4 & 3\end{vmatrix} = -3$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$\therefore \text{adj. A} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj. A} = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Cramer's Rule

Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1, a_{21}x + a_{22}y + a_{23}z = b_2, \qquad a_{31}x + a_{32}y + a_{33}z = b_3$$

Then $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$,

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

- If $\Delta \neq 0$, then $x = \frac{\Delta_1}{\Delta}, \; y = \frac{\Delta_2}{\Delta}, \; z = \frac{\Delta_3}{\Delta}$
- If $\Delta = 0$ and at least oneof $\Delta_1, \Delta_2, \Delta_3$then system is inconsistent i.e. has no solution.
- If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then system has infinitely many solutions. Take any two equations out of the three given equations and shift one of the variables, say z, on the right hand side. Solve these two equations by Cramer's rule to obtain x, y in terms of z.

Q1. Solve by Cramer's rule$5x - 7y + z = 11, 6x - 8y - z = 15, \; 3x + 2y - 6z = 7$

Sol. By Cramer's rule$\Delta = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix} = 5(48 + 2) + 7(-36 + 3) + 1(12 + 18) = 55$

$$\Delta_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix} = 11(48 + 2) + 7(-90 + 7) + 1(30 + 56) = 55$$

$$\Delta_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix} = 5(-90 + 7) - 11(-36 + 3) + 1(42 - 45) = -55$$

$$\Delta_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix} = 5(-56-30) + 7(42-45) + 11(12+24) = -55$$

$$x = \frac{\Delta_1}{\Delta} = \frac{55}{55} = 1, \quad y = \frac{\Delta_2}{\Delta} = \frac{-55}{55} = -1, \quad z = \frac{\Delta_3}{\Delta} = \frac{-55}{55} = -1$$

Q2. Solve by Cramer's rule $2x - y + z = 4, x + 3y + 2z = 12, \; 3x + 2y + 3z = 10$

Sol. By Cramer's rule $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix} = 2(9-4) + 1(3-6) + 1(2-9) = 0$

$$\Delta_1 = \begin{vmatrix} 4 & -1 & 1 \\ 12 & 3 & 2 \\ 10 & 2 & 3 \end{vmatrix} = 4(9-4) + 1(36-20) + 1(24-30) = 30 \neq 0$$

Therefore system is inconsistent and has no solution.

Matrix Inversion Method

Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1, a_{21}x + a_{22}y + a_{23}z = b_2, \qquad a_{31}x + a_{32}y + a_{33}z = b_3$$

Then $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

- If $|A| \neq 0$, then system has unique solution.

The system can be written as $AX = B \xrightarrow{\text{yields}} X = A^{-1}B$

- If $|A| = 0$ and $(\text{adj}. A)B = 0$, then system is consistent and has infinitely many solutions.
- If $|A| = 0$ and $(\text{adj}. A)B \neq 0$, then system is inconsistent.

Q3. Solve by matrix inversion method $x + 2y + z = 7, \; x + 3z = 11, \; 2x - 3y = 1$

Sol. Here $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$|A| = 1(0+9) - 2(0-6) + 1(-3-0) = 18 \neq 0$. Thus system has unique solution.

Cofactors of elements a_{ij} in A are

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} = 9$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -3$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = -3$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 7$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$\therefore \text{adj. A} = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj. A} = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore x = 2, y = 1, z = 3$

Rank: A matrix **A** is said to be of rank r if

(i) All the minors, in **A**, of order greater than r are zero.

(ii) There exists atleast one minor of order r in **A** which is non zero.

Q. Find rank of $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ 3 & -3 & 1 \end{vmatrix} = 1(-3) - 1(2) + 0 = -5 \neq 0 \therefore \rho(A) = 3$$

Q. What is the rank of a non-singular matrix of order n?

Sol. The rank of a non-singular matrix of order n is n because the determinant of non-singular matrix A is non-zero.

Q. If A is a non- zero row and B is a non- zero column matrix, show that rank AB =1.

Sol. Let A = $[y_1 \quad y_2 \quad \cdots \cdots \quad y_n]$ and B = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$

Then AB = $[x_1y_1 + x_2y_2 + \ \ldots \ldots . + x_ny_n]$ which is a singleton matrix. Hence rank AB = 1.

Linearly Dependent vectors: A set of vectors $X_1, X_2, \ldots \ldots .., X_n$ is said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots \ldots ., \alpha_n$, atleast one α_i non- zero, such that

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n = 0$$

e.g. $X_1 = (2,4)$ and $X_2 = (1,2)$, then $X_1 + (-2)X_2 = 0$. Therefore X_1 and X_1 are linearly dependent vectors.

Linearly Independent vectors: A set of vectors $X_1, X_2, \ldots \ldots .., X_n$ is said to be linearly independent If for scalars $\alpha_1, \alpha_2, \ldots \ldots ., \alpha_n$,

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n = 0$$

Implies all α_i are zero.

e.g. $X_1 = (2,0)$ and $X_2 = (0,4)$,

then $\alpha_1 X_1 + \alpha_2 X_2 = 0 \xrightarrow{\text{yields}} (2\alpha_1, 4\alpha_2) = 0 \xrightarrow{\text{yields}} \alpha_1 = 0$ and $\alpha_2 = 0$. Therefore X_1 and X_2 are linearly independent vectors.

Q. Determine whether the set $\{(3,2,4),\ (1,0,2),\ (1,-1,-1)\}$ of vectors linearly independent.

Sol. $\begin{vmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{vmatrix} = 3(0+2) - 2(-1-2) + 4(-1) = 4 \neq 0$

∴ Vectors linearly independent.

Q. Determine whether the set $\{(2,2,1),\ (1,-1,1),\ (1,0,1)\}$ of vectors linearly independent.

Sol. $\begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2(-1-0) - 2(1-1) + 1(0+1) = -1 \neq 0$

∴ Vectors linearly independent.

Conditions for consistency of system of equations

Non-homogenous system of linear equations (AX=B)

i. If $\rho(A{:}B) = \rho(A) = \text{number of unknowns}$, the system has unique solution.
ii. If $\rho(A{:}B) = \rho(A) < \text{number of unknowns}$, the system has an infinite number of solutions.
iii. If $\rho(A{:}B) \neq \rho(A)$, the system is inconsistent i.e. it has no solution.

Homogenous system of linear equations (AX=0)

i. This system always has a solution X=0 called the null or trivial solution.
ii. If $\rho(A) = \text{number of unknowns}$, the system has unique solution i.e. trivial solution.
iii. If $\rho(A) < \text{number of unknowns}$, the system has an infinite number of solutions.

Gauss Elimination Method to solve system of equations:

i. Convert the system to matrix form.
ii. Convert the matrix to Echelon form (by applying row operations only).
iii. Apply back substitution i.e. convert the matrix to system of equations.

Gauss Jordan Method to solve system of equations:

i. Convert the system to matrix form.
ii. Convert the matrix to Normal form (by applying row operations only).
iii. Apply back substitution i.e. convert the matrix to system of equations.

Q. Solve the system of equations $x + y + z = 3,\ \ 3x - 9y + 2z = -4,\ \ 5x - 3y + 4z =$

Sol. $[A: B] = \begin{bmatrix} 1 & 1 & 1: 3 \\ 3 & -9 & 2:-4 \\ 5 & -3 & 4: 6 \end{bmatrix}$ (Operating $R_2 \to R_2 - 3R_1,\ \ R_3 \to R_3 - 5R_1$)

$\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 1 & 1 : 3 \\ 0 & -12 & -1:-13 \\ 0 & -8 & -1: -9 \end{bmatrix}$ (Operating $R_2 \to \frac{R_2}{-12}$)

$\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 1 & 1 : 3 \\ 0 & 1 & \frac{1}{12} : \frac{13}{12} \\ 0 & -8 & -1 : -9 \end{bmatrix}$ (Operating , $R_3 + 8R_2$) $\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 1 & 1 : 3 \\ 0 & 1 & \frac{1}{12} : \frac{13}{12} \\ 0 & 0 & \frac{-4}{12} : \frac{-4}{12} \end{bmatrix}$

Since $\rho[A: B] = \rho(A) = \text{number of unknowns}$, so system is consistent and has unique solution.

Applying back substitution, from R_3, $\frac{-4}{12} z = \frac{-4}{12} \xrightarrow{\text{yields}} z = 1$

From $R_2, y + \frac{1}{12} z = \frac{13}{12} \xrightarrow{\text{yields}} y = \frac{13}{12} - \frac{1}{12} z \xrightarrow{\text{yields}} y = \frac{12}{12} = 1$

From $R_1,\ x + y + z = 3 \xrightarrow{\text{yields}} x + 1 + 1 = 3 \xrightarrow{\text{yields}} x = 1$

Hence solution is $x = 1,\ \ y = 1,\ \ z = 1$

Q. Solve the system of equations

$x + 2y\ \ + z = 2,\ \ 3x + y - 2z = 1,\ \ 4x - 3y - z = 3,\ \ 2x + 4y + 2z = 4$

Sol. $[A: B] = \begin{bmatrix} 1 & 2 & 1 : & 2 \\ 3 & 1 & -2: & 1 \\ 4 & -3 & -1: & 3 \\ 2 & 4 & 2 : & 4 \end{bmatrix}$ (Operating $R_2 - 3R_1, R_3 - 4R_1, R_4 - 2R_1$)

$\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 2 & 1 : & 2 \\ 0 & -5 & -5: & -5 \\ 0 & -11 & -5: & -5 \\ 0 & 0 & 0 : & 0 \end{bmatrix}$ (Operating $R_2 \to \frac{R_2}{-5}$)

$\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 2 & 1 : & 2 \\ 0 & 1 & 1 : & 1 \\ 0 & -11 & -5: & -5 \\ 0 & 0 & 0 : & 0 \end{bmatrix}$ (Operating , $R_3 + 11R_2$) $\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 2 & 1: & 2 \\ 0 & 1 & 1: & 1 \\ 0 & 0 & 6: & 6 \\ 0 & 0 & 0: & 0 \end{bmatrix}$

Since $\rho[A: B] = \rho(A) = \text{number of unknowns}$, so system is consistent and has unique solution.

Applying back substitution, from R_3, $6z = 6 \xrightarrow{\text{yields}} z = 1$

From $R_2, y + z = 1 \xrightarrow{\text{yields}} y = 1 - z \xrightarrow{\text{yields}} y = 1 - 1 = 0$

From R_1, $x + 2y + z = 2 \xrightarrow{\text{yields}} x + 0 + 1 = 2 \xrightarrow{\text{yields}} x = 1$

Hence solution is $x = 1, y = 0, z = 1$

Q. Find the values of α for unique solution and infinitely many solutions. Hence solve the system in each case. $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \alpha z = -3$

Sol. $[A:B] = \begin{bmatrix} 3 & -1 & 4: & 3 \\ 1 & 2 & -3: & -2 \\ 6 & 5 & \alpha: & -3 \end{bmatrix}$ (Operating $R_2 \leftrightarrow R_1$)

$\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 2 & -3: & -2 \\ 3 & -1 & 4: & 3 \\ 6 & 5 & \alpha: & -3 \end{bmatrix}$ (Operating $R_2 - 3R_1, R_3 - 6R_1$)

$\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 2 & -3: & -2 \\ 0 & -7 & 13: & 9 \\ 0 & -7 & \alpha + 18: & 9 \end{bmatrix}$ (Operating , $R_3 - R_2$)

$\xrightarrow{\text{yields}} \begin{bmatrix} 1 & 2 & -3: & -2 \\ 0 & -7 & 13: & 9 \\ 0 & 0 & \alpha + 5: & 0 \end{bmatrix}$

(i) system is consistent and has unique solution if $\rho[A:B] = \rho(A) = \text{number of unknowns}$

i.e. $\alpha + 5 \neq 0 \xrightarrow{\text{yields}} \alpha \neq -5$

When $\alpha + 5 \neq 0$, $[A:B] = \begin{bmatrix} 1 & 2 & -3: & -2 \\ 0 & -7 & 13: & 9 \\ 0 & 0 & \alpha + 5: & 0 \end{bmatrix}$

Applying back substitution, from R_3, $(\alpha + 5)z = 0 \xrightarrow{\text{yields}} z = 0$

From R_2, $-7y + 13z = 9 \xrightarrow{\text{yields}} -7y = 9 \xrightarrow{\text{yields}} y = \frac{-9}{7}$

From R_1, $x + 2y - 3z = -2 \xrightarrow{\text{yields}} x - \frac{18}{7} = -2 \xrightarrow{\text{yields}} x = \frac{4}{7}$

Hence solution is $x = \frac{4}{7}, y = \frac{-9}{7}, z = 0$

(ii) system is consistent and has infinitely many solutions if

$\rho[A:B] = \rho(A) <$ number of unknowns   i.e. $\alpha + 5 = 0 \xrightarrow{\text{yields}} \alpha = -5$

When $\alpha + 5 = 0$, $[A:B] = \begin{bmatrix} 1 & 2 & -3: & -2 \\ 0 & -7 & 13: & 9 \\ 0 & 0 & 0: & 0 \end{bmatrix}$

Let $z = t$ Applying back substitution,

From R_2, $-7y + 13z = 9 \xrightarrow{\text{yields}} -7y = 9 - 13t \xrightarrow{\text{yields}} y = \frac{13t-9}{7}$

From R_1, $x + 2y - 3z = -2 \xrightarrow{\text{yields}} x + \frac{26t-18}{7} - 3t = -2 \xrightarrow{\text{yields}} x + \frac{5t-18}{7} = -2$

$\xrightarrow{\text{yields}} x = \frac{4-5t}{7}$

Hence solution is $x = \frac{4-5t}{7}, y = \frac{13t-9}{7}, z = t$

Orthogonal matrix: A square matrix A is said to be orthogonal if

$$AA^T = A^TA = I.$$

$$\text{e.g. } A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Unitary matrix: A square matrix A is said to be Unitary if

$$A^\theta A = AA^\theta = I$$

where $A^\theta = (\overline{A})^T$

$$\text{e.g. } A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Note: Every orthogonal matrix is unitary.

Hermitian matrix: A square matrix A is said to be Hermitian matrix if

$A^\vartheta = A$ i.e. $a_{ij} = \overline{a_{ji}}$

Diagonal elements of a Hermitian matrix are real numbers.

$$\text{e.g. } A = \begin{bmatrix} 1 & 2+3i & 5-6i \\ 2-3i & 2 & 9-6i \\ 5+6i & 9+6i & -11 \end{bmatrix}$$

Skew Hermitian matrix: A square matrix A is said to be skew Hermitian matrix if

$A^{\vartheta} = -A$ i.e. $a_{ij} = -\overline{a_{ji}}$

Diagonal elements of a skew Hermitian matrix are either zero or purely imaginary numbers.

$$\text{e.g. } A = \begin{bmatrix} 1 & 2+3i & -5-6i \\ -2+3i & 2 & -9+6i \\ 5-6i & 9+6i & -11 \end{bmatrix}$$

Similar matrices: A square matrix A is said to be similar to a square matrix B if there exists an invertible matrix P such that $A = P^{-1}BP$. P is called similarity matrix. This relation of similarity is a symmetric relation.

Cayley Hamilton theorem: Every square matrix satisfies its own characteristic equation.

Eigen values and Eigen Vectors: Let A be a square matrix. Then the equation determinant $(A - \alpha I)$ =0 is called characteristic equation of A. The roots of characteristic equation of A are called Eigen values or latent roots of matrix A.

A column vector X satisfying the equation $AX = \alpha X$ i.e. $(A - \alpha I)X = 0$ is called Eigen vector or latent vector of matrix A corresponding to eigen value α.

Diagonalizable matrix: A square matrix A is said to be diagonalizable if there exists an invertible matrix P such that

$$P^{-1}BP = D$$

Where D is a diagonal matrix and the diagonal elements of D are Eigen values of A.

1. The characteristics equation of a matrix A is $t^2 - t - 1 = 0$, then determine A^{-1}.

Sol. By Cayley Hamilton theorem, every square matrix satisfies its characteristic equation.

Therefore A^2-A-1=0

or A^2-A=1

Premultipying both sides by A

$A - I = A^{-1}$

2. Prove eigen value of a Hermitian matrix is real.

Sol. Let A be a Hermitian matrix. Therefore $A^{\theta} = A$ − − − − − −(1)

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$$AX = \alpha X \xrightarrow{\text{yields}} (AX)^{\theta} = (\alpha X)^{\theta} \xrightarrow{\text{yields}} X^{\theta}A^{\theta} = \overline{\alpha}X^{\theta} \xrightarrow{\text{yields}} X^{\theta}A = \overline{\alpha}X^{\theta} \quad \text{(using (1))}$$

Post multiplying both sides by X, we get

$$X^{\theta}(AX) = \overline{\alpha}(X^{\theta}X) \xrightarrow{\text{yields}} X^{\theta}\,\alpha X = \overline{\alpha}(X^{\theta}X) \xrightarrow{\text{yields}} \alpha(X^{\theta}X) = \overline{\alpha}(X^{\theta}X) \xrightarrow{\text{yields}} \alpha = \overline{\alpha}$$

Hence α is a real number. Therefore Eigen value of a Hermitian matrix is real.

1. Prove $\frac{|A|}{\alpha}$ is an eigen value of adj (A)eigen vector remaining the same if α is an eigen value of A and X is corresponding Eigen vector.

Sol. Let A be a square matrix$- - - - - -$ (1)

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$AX = \alpha X$ (using (1))

Pre- multiplying both sides by adj (A), we get

$$\text{adj }(A)(AX) = \text{adj }(A)\alpha X \xrightarrow{\text{yields}} (\text{adj }(A)A)X = \alpha(\text{adj }(A)X) \xrightarrow{\text{yields}} |A|X = \alpha(\text{adj }(A)X)$$

$$\text{adj }(A)X = \frac{|A|}{\alpha}X$$

Hence $\frac{|A|}{\alpha}$ is an eigen value of adj (A) and X is corresponding Eigen vector.

2. Prove that product of two orthogonal matrices is orthogonal matrix

Sol. Let A and B be two orthogonal matrices. Therefore

$AA^T = A^TA = I$ and $BB^T = B^TB = I$

Now $(AB)(AB)^T = ABB^TA^T = AIA^T = AA^T = I$ and

$(AB)^T(AB) = B^TA^T AB = BIB^T = BB^T = I$

Hence AB is an orthogonal matrix. Therefore product of two orthogonal matrices is orthogonal matrix.

3. Prove that transpose of an orthogonal matrix is orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Now $A^T(A^T)^T = A^TA = I$ and

$(A^T)^TA^T = AA^T = I$

Hence A^T is an orthogonal matrix

Therefore transpose of an orthogonal matrix is orthogonal matrix.

4. Prove that inverse of an orthogonal matrix is an orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Now $A^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1} = (A^TA)^{-1} = I^{-1} = I$ and

$(A^{-1})^TA^{-1} = (A^T)^{-1}A^{-1} = (AA^T)^{-1} = I^{-1} = I$

Hence A^{-1} is an orthogonal matrix

Therefore inverse of an orthogonal matrix is orthogonal matrix.

5. Prove that determinant of an orthogonal matrix is ± 1.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Taking determinant on both sides

$|AA^T| = |I| \xrightarrow{\text{yields}} |A||A^T| = 1 \xrightarrow{\text{yields}} |A||A| = 1 \xrightarrow{\text{yields}} |A|^2 = 1 \xrightarrow{\text{yields}} |A| = \pm 1$

(Because $|CD| = |C||D|$, $|I| = 1$, $|A| = |A^T|$)

6. Prove that inverse of a unitary matrix is an unitary matrix.

Sol. Let A be unitary matrix. Therefore

$A^\theta A = AA^\theta = I$ where $A^\theta = (\overline{A})^T$

Now $A^{-1}(A^{-1})^\theta = A^{-1}(A^\theta)^{-1} = (A^\theta A)^{-1} = I^{-1} = I$ and

$(A^{-1})^\theta A^{-1} = (A^\theta)^{-1}A^{-1} = (AA^\theta)^{-1} = I^{-1} = I$

Hence A^{-1} is an orthogonal matrix

Therefore inverse of an orthogonal matrix is orthogonal matrix.

7. State and prove Cayley Hamilton theorem.

Sol. Statement: Every square matrix satisfies its own characteristic equation.

Proof: Let A be a square matrix of order n and its characteristic equation be $|A - \lambda I| = 0$

i.e. $(-1)^n\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots \ldots \ldots + a_n = 0$

Required to be proved: $(-1)^nA^n + a_1A^{n-1} + a_2A^{n-2} + \cdots \ldots \ldots + a_nI = 0$

Here λ is an eigen value of A.

$[A - \lambda I]$ is a matrix of order n $\xrightarrow{\text{yields}}$ $\text{adj.}(A - \lambda I)$ is a matrix of order (n-1).

Therefore we can write $\text{adj.}(A - \lambda I) = P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots \ldots \ldots + P_n$ where $P_1, P_2, \ldots \ldots \ldots P_n$ are square matrices.

Also $A(\text{adj.}A) = |A|I$ $\xrightarrow{\text{yields}}$ $(A - \lambda I)\text{adj.}(A - \lambda I) = |A - \lambda I|I$

$\xrightarrow{\text{yields}}$ $(A - \lambda I)[P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots \ldots \ldots + P_n] = [(-1)^n\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots \ldots \ldots + a_n]I$

Comparing coefficients of like powers of A, we get

$$-P_1 = (-1)^n I$$

$$AP_1 - P_2 = a_1 I$$

$$AP_2 - P_3 = a_2 I$$

$$AP_3 - P_4 = a_3 I$$

……………………. (and so on)

$$AP_{n-1} - P_n = a_{n-1} I$$

$$AP_n = a_n I$$

Pre-multiplying these equations by $A^n, A^{n-1}, A^{n-2}, \ldots \ldots \ldots, A, I$ respectively on both sides and adding, we get $0 = (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots \ldots \ldots + a_n I$

$\xrightarrow{\text{yields}} (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots \ldots \ldots + a_n I = 0$

(Hence proved).

8. Find characteristic equation of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{\text{yields}} \begin{vmatrix} 1-\alpha & 0 & -1 \\ 1 & 2-\alpha & 1 \\ 2 & 2 & 3-\alpha \end{vmatrix} = 0$

$\xrightarrow{\text{yields}} \alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$

9. Is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ diagonalizable?

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{\text{yields}} \begin{vmatrix} 1-\alpha & 0 & 0 \\ 0 & 3-\alpha & -1 \\ 0 & -1 & 3-\alpha \end{vmatrix} = 0$

$\xrightarrow{\text{yields}} \alpha^3 - 7\alpha^2 + 14\alpha - 8 = 0 \xrightarrow{\text{yields}} \alpha = 1,2,4$

Since A has three distinct Eigen values, ∴ it has three linearly independent Eigen vectors. Hence A

A is diagonalizable.

10. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. Hence find A^{-1}. Also find Eigen values and vectors of A

Sol. $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{\text{yields}} \begin{vmatrix} 1-\alpha & 4 \\ 3 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{\text{yields}} \alpha^2 - 3\alpha - 10 = 0 \xrightarrow{\text{yields}} \alpha = -2,5$

By Cayley Hamilton theorem $A^2 - 3A - 10I = 0$ ……………………….(*)

Now $A^2 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix}$,

$\therefore A^2 - 3A - 10I = \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix} + \begin{bmatrix} -3 & -12 \\ -9 & -6 \end{bmatrix} + \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ Cayley Hamilton theorem is verified for given matrix A.

Multiplying both sides of (*) by A^{-1}, we get $A - 3I = 10A^{-1} \xrightarrow{\text{yields}} A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

Let $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

$\therefore [A - \alpha I]X_1 = 0 \xrightarrow{\text{yields}} [A - (-2)I]X_1 = 0 \xrightarrow{\text{yields}} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\xrightarrow{\text{yields}} 3x + 4y = 0, 3x + 4y = 0 \xrightarrow{\text{yields}} \frac{x}{-4} = \frac{y}{3}$

$\therefore X_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

Let $X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$\therefore [A - \alpha I]X_2 = 0 \xrightarrow{\text{yields}} [A - (5)I]X_2 = 0 \xrightarrow{\text{yields}} \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\xrightarrow{\text{yields}} -4x + 4y = 0, 3x - 3y = 0 \xrightarrow{\text{yields}} x = y \xrightarrow{\text{yields}} \frac{x}{1} = \frac{y}{1}$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

11. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence find A^{-1}.

Sol. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{\text{yields}} \begin{vmatrix} 2-\alpha & -1 & 1 \\ -1 & 2-\alpha & -1 \\ 1 & -1 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{\text{yields}} \alpha^3 - 6\alpha^2 + 9\alpha - 4 = 0$

By Cayley Hamilton theorem $A^3 - 6A^2 + 9A - 4I = 0$ …………………..(i)

L.H.S. $A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$$A^3 = AAA = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Hence $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton theorem is verified for the given matrix A

From(i), $4I = A^3 - 6A^2 + 9A$

Multiplying both sides by A^{-1}, we get

$$A^{-1} = \frac{1}{4}[A^2 - 6A + 9I] = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

12. Find Eigen values and vectors of $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 3-\alpha & 1 & -1 \\ -2 & 1-\alpha & 2 \\ 0 & 1 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0 \xrightarrow{yields} \alpha = 1,2,3$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\xrightarrow{yields} 2x + y - z = 0\,, -2x + 2z = 0, y + z = 0$

From first two equations, $\frac{x}{\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix}} \xrightarrow{yields} \frac{x}{2} = \frac{y}{-2} = \frac{z}{2} \xrightarrow{yields} \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$

$\therefore\ X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

$$\therefore\ [A - \alpha I]X_2 = 0 \xrightarrow{yields} [A - (2)I]X_2 = 0 \xrightarrow{yields} \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} x + y - z = 0\,, -2x - y + 2z = 0, y = 0$$

From first two equations, $\frac{x}{\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}} \xrightarrow{yields} \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$

$\therefore\ X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 3$.

$$\therefore\ [A - \alpha I]X_3 = 0 \xrightarrow{yields} [A - (3)I]X_3 = 0 \xrightarrow{yields} \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} y - z = 0\,, -2x - 2y + 2z = 0, y - z = 0 \xrightarrow{yields} y - z = 0\,, -2x - 2y + 2z = 0$$

$\therefore$ we get , $\frac{x}{\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -1 & 0 \\ 2 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix}} \xrightarrow{yields} \frac{x}{0} = \frac{y}{2} = \frac{z}{2} \xrightarrow{yields} \frac{x}{0} = \frac{y}{1} = \frac{z}{1}$

$\therefore\ X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

13. Find Eigen values and vectors of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 1-\alpha & 1 & 0 \\ 0 & 1-\alpha & 1 \\ 0 & 0 & 1-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} (1-\alpha)^3 \xrightarrow{yields} \alpha = 1,1,1$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{yields} y = 0\,, z = 0.\ Take\ x = 1$$

$\therefore\ X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

14. Examine whether the following matrix is diagonalizable. If so, obtain the matrix P such that $P^{-1}AP$ is a diagonal matrix. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Sol. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} -2-\alpha & 2 & -3 \\ 2 & 1-\alpha & -6 \\ -1 & -2 & 0-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} -(\alpha+3)(\alpha+3)(\alpha-5) = 0 \xrightarrow{yields} \alpha = -3, -3, 5$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = -3$.

$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (-3)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1$)

$\xrightarrow{yields} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{yields} x + 2y - 3z = 0$

Choose $y = 0 \xrightarrow{yields} x - 3z = 0 \xrightarrow{yields} \frac{x}{3} = \frac{z}{1}$

$\therefore\ X_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is the first Eigen vector of A corresponding to Eigen value $\alpha = -3$.

Choose $z = 0 \xrightarrow{yields} x + 2y = 0 \xrightarrow{yields} \frac{x}{-2} = \frac{y}{1}$

$\therefore\ X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is another Eigen vector of A corresponding to Eigen value $\alpha = -3$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$\therefore\ [A - \alpha I]X_3 = 0 \xrightarrow{yields} [A - (5)I]X_3 = 0 \xrightarrow{yields} \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\xrightarrow{yields} -7x + 2y - 3z = 0, 2x - 4y - 6z = 0, -x - 2y - 5z = 0$

$\therefore$ from first two equations we get , $\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -3 & -7 \\ -6 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}} \xrightarrow{yields} \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24} \xrightarrow{yields} \frac{x}{1} = \frac{y}{12} = \frac{z}{-1}$

$\therefore\ X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$\therefore$ Modal Matrix $P = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

$$|P| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix} = 8 \neq 0$$. Hence vectors are linearly independent and the given matrix is

Diagonalizable.

$$P^{-1} = \frac{Adj.P}{|P|} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$$

Diagonal Matrix = D = $P^{-1}AP = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

$$= \frac{1}{8}\begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

15. Let T be a linear transformation defined by $T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Find $T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right]$.

Sol.The matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent and hence form a basis in the space of 2×2 matrices. We write for any scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero

$$\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \alpha_1\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}$$

Comparing the elements and solving the resulting system of equations, we get $\alpha_1 = 4, \alpha_2 = 1, \alpha_3 = -2, \alpha_4 = 5$. Since T is a linear transformation,

$$\therefore T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right] = \alpha_1 T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_2 T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] + \alpha_3 T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] + \alpha_4 T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= 4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 20 \\ 36 \end{pmatrix}$$