# Volume Of Solids Of Revolution

## Cartesian

$$y = f(x)$$

or $f(x,y) = c$

Revolution about x-axis

$$V_x = \int_{a_1}^{b_1} \pi y^2 \, dx$$

y-axis

$$V_y = \int_{a_2}^{b_2} \pi x^2 \, dy$$

$$V_{x=a} = \int_{a_3}^{b_3} \pi (x-a)^2 \, dy$$

$$V_{y=b} = \int_{a_4}^{b_4} \pi (y-b)^2 \, dx$$

## Parametric

$$x = f(t)$$

$$y = g(t)$$

$$V_x = \int_{a_1}^{b_1} \pi (g(t))^2 \cdot \frac{dx}{dt} \cdot dt$$

or $\displaystyle\int_{a_1}^{b_1} \pi (g(t))^2 \frac{df}{dt} \, dt$

Similarly

$$V_y = \int_{a_2}^{b_2} \pi [f(t)]^2 \frac{dg}{dt} \, dt$$

## Polar

$$r = f(\theta)$$

or $f(r,\theta) = c$

$\theta = 0 \rightarrow$ Initial axis

$\theta = \pi/2 \rightarrow$ Axis perpendicular to initial axis

Revolution about line $\theta = 0$ (Initial line)

$$V_{\theta=0} = \int_{\theta=\alpha}^{\beta} \frac{2}{3} \pi r^3 \sin\theta \, d\theta$$

Revolution about line $\theta = \pi/2$ (Line perpendicular to initial line)

$$V_{\theta=\pi/2} = \int_{\theta=\alpha_1}^{\beta_1} \frac{2}{3} \pi r^3 \cos\theta \, d\theta$$

RESULT :

$$\int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{(n-1)(n-3)\cdots \text{go on decreasing by } 2}{n(n-2)\cdots \text{go on decreasing by } 2} \times \begin{cases} \frac{\pi}{2} & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$$\int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{(n-1)(n-3)\cdots \text{go on decreasing by } 2}{n(n-2)\cdots \text{go on dec. by } 2} \times \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ \text{otherwise not} \end{cases}$$

$$\int_0^{\pi/2} \cos^m\theta \cdot \sin^n\theta \, d\theta = \frac{(m-1)(m-3)\cdots\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots\cdots} \times \left\{ \pi/2 \text{ if both } m \text{ and } n \text{ are even, otherwise not} \right\}$$

Revolution of Circle about x-axis / Initial axis

**Cartesian**

$$x^2 + y^2 = a^2 \quad \text{①}$$


$$V_x = \int_{-a}^{a} \pi y^2 \, dx$$

$$= 2\int_0^a \pi (a^2 - x^2)\, dx$$

$$= 2\pi \int_0^a a^2 - x^2 \, dx$$

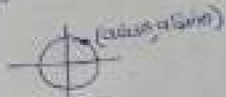
$$= 2\pi \cdot \left[ a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[ a^3 - \frac{a^3}{3} \right] = 2\pi \cdot \frac{2a^3}{3}$$

$$= \frac{4}{3}\pi a^3$$

**Parametric**

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \end{aligned}\right\} \text{②}$$




$$V_x = \int_0^{\pi} \pi a^2 \sin^2\theta \cdot \frac{dx}{d\theta}\, d\theta$$

$$= \pi a^2 \int_0^{\pi} \sin^2\theta \cdot (-a\sin\theta)\, d\theta$$

$$= -\pi a^3 2 \int_0^{\pi/2} \sin^3\theta \cdot d\theta$$

$$= -2\pi a^3 \frac{2}{3 \cdot 1}$$

$$|V_x| = \frac{4}{3}\pi a^3$$

* In these cases Revolution taken about y-axis / axis perpendicular to initial axis will generate Same Volume.

**polar form**

$$r = a \quad \text{③}$$


Equations ①, ② & ③ represents Same Circle

$$V_{\theta=0} = \int_0^{\pi} \frac{2}{3}\pi r^3 \cdot \sin\theta \, d\theta$$

$$= \int_0^{\pi} \frac{2}{3}\pi a^3 \sin\theta \, d\theta$$

$$\frac{2}{3}\pi a^3 \left[ -\cos\theta \right]_0^{\pi}$$

$$= \frac{2}{3}\pi a^3 \left[ -(-1) - (-1) \right]$$

$$\frac{2}{3}\pi a^3 \cdot 2$$

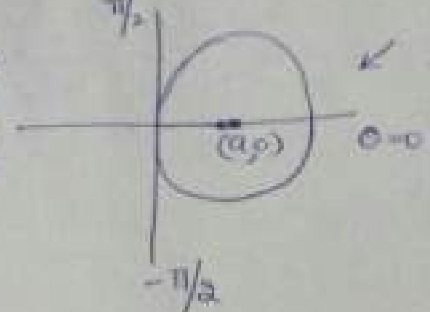
$$V_{\theta=0} \,(\text{initial axis}) = \frac{4}{3}\pi a^3$$

polar | Cartesian

1. $r = 2a\cos\theta$ or $(x-a)^2 + y^2 = a^2$
2. $r = 2a\sin\theta$ or $x^2 + (y-a)^2 = a^2$

] Equations of circle.

1. $r = 2a\cos\theta$




Revolution about $x$-axis or $\theta = 0$ will generate sphere but not about $y$-axis

$$V_{\theta=0} = \int_0^{\pi/2} \frac{2}{3}\pi r^3 \sin\theta \, d\theta = \int_0^{\pi/2} -\frac{2}{3}\pi \; 8a^3\cos^3\theta \, {}_-\sin\theta \, d\theta$$

$$= -\frac{2}{3}\pi \cdot 8a^3 \int_0^{\pi/2} \cos^3\theta \cdot \sin\theta \cdot d\theta$$

$$= -\frac{2}{3}\pi \cdot 8a^3 \cdot \left.\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$\left[ \int [f(x)]^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c \right]$$

$$= -\frac{2}{3}\pi \cdot \cancel{8}^{2} a^3 \cdot -\frac{1}{\cancel{4}}$$

$$= \frac{4}{3}\pi a^3$$

Revolution about $y$-axis

$$V_{\theta=\frac{\pi}{2}} = \int_{-\pi/2}^{\pi/2} \frac{2}{3}\pi r^3 \cos\theta \, d\theta = 2\int_0^{\pi/2} \frac{2}{3}\pi \, 8a^3 \cos^4\theta \, d\theta$$

(axis perpendicular to initial axis)

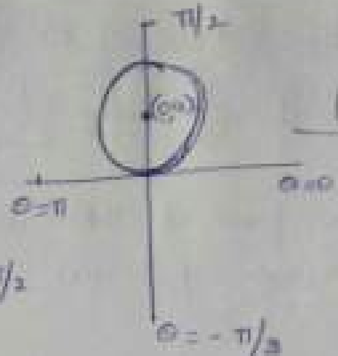
$$= \frac{\cancel{32}^{4} \pi a^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}$$

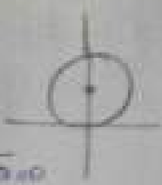
$$\left[ \because f(x) = f(a - x) \Rightarrow \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \right]$$

$$\boxed{V_{\theta=\pi/2} = 2\pi^2 a^3}$$

(2) $r = 2a\sin\theta$

$x^2+(y-a)^2 = a^2$



Revolution about $y$-axis or $\theta=\pi/2$ will generate sphere but not about $x$-axis

$$V_{\theta=\pi/2} = \int_0^{\pi/2} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{2}{3}\pi \cdot 8a^3 \sin^3\theta \cdot \cos\theta \, d\theta$$

$$= \frac{16}{3}\pi a^3 \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{16}{3}\pi a^3 \cdot \frac{1}{4} = \frac{4}{3}\pi a^3$$

$$\boxed{V_{\theta=\pi/2} = \frac{4\pi a^3}{3}}$$

Revolution about $x$-axis

$$V_{\theta=0} = \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta = \int_0^{\pi} \frac{2}{3}\pi \, 8a^3 \sin^3\theta \cdot \sin\theta \, d\theta$$

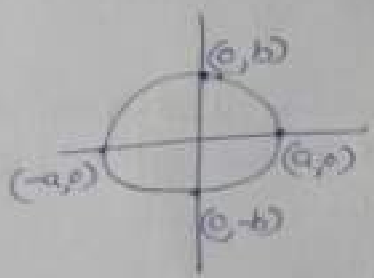
$$= 2\int_0^{\pi/2} \frac{16}{3}\pi a^3 \sin^4\theta \, d\theta = \frac{32\pi a^3}{3} \cdot \frac{\cancel{3}}{4\cdot 2} \cdot \frac{\pi}{2}$$

$$\boxed{V_{\theta=0} = 2\pi^2 a^3}$$

* Find the Volume of the Solid formed by revolving $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ along $x$-axis $\rightarrow$ ①

Parametric form of ellipse is

$x = a\cos\theta, \quad y = b\sin\theta$




Sol Cartesian

$$V_x = \int_{-a}^{a} \pi y^2\, dx = 2\int_{0}^{a} \pi b^2\left(1-\frac{x^2}{a^2}\right)\cdot dx$$

$$= 2\pi b^2 \left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi b^2\left[a - \frac{1a}{3}\right] = \frac{4}{3}\pi a b^2$$

$$V_y = \int_{-b}^{b} \pi x^2\, dy = \frac{4}{3}\pi a^2 b$$

$$V_x = \int_0^{\pi} \pi y^2 \frac{dx}{d\theta}\, d\theta$$

Parametric: $V_x = \int_0^{\pi} \pi\, b^2\sin^2\theta \cdot -a\sin\theta \cdot d\theta$

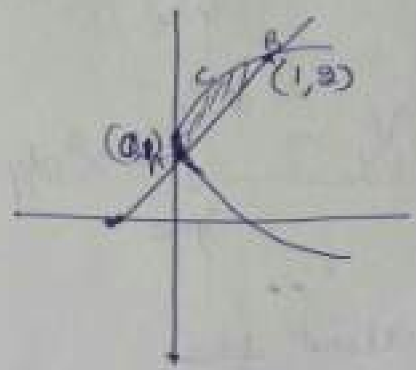
$$= \pi a b^2 \cdot 2\int_0^{\pi/2} \sin^3\theta \cdot d\theta$$

$$= \pi a b^2 \cdot 2 \cdot \frac{2}{3\cdot 1} = \frac{4}{3}\pi a b^2$$

Similarly $V_y = \int_{-\pi/2}^{\pi/2} \pi x^2 \cdot \frac{dy}{d\theta}\cdot d\theta = \frac{4}{3}\pi a^2 b$

d the Volume of the Solid generated by revolving the region bounded by the Curves $y = 1+\sqrt{x}$ and $y = 1+x$ about $y$-axis

Sol.
$$y - 1 = \sqrt{x}$$
$$\text{or} \quad (y-1)^2 = x$$




$$y = 1+x$$
$$x - y = -1$$
$$\text{or} \quad \frac{x}{-1} + \frac{y}{1} = 1$$

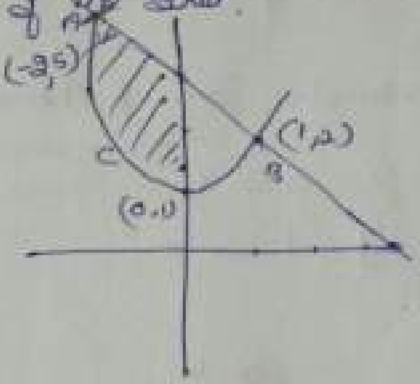
Required Volume = (Volume generated by line about y-axis) − (Volume generated by $\widehat{ACB}$ about y-axis)

$$= \underset{\text{over } AB}{\int_1^2 \pi x^2 dy} - \underset{\text{over } \widehat{ACB}}{\int_1^2 \pi x^2 dy}$$

$$= \int_1^2 \pi (y-1)^2 dy - \int_1^2 \pi (y-1)^4 . dy$$

Required $V = \frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}$ Cubic unit

The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about $x$-axis to generate a solid. Find the volume of the solid.




$x^2 = y-1$ — ①

$\frac{x}{3}+\frac{y}{3} = 1$ — ②

Required volume

$$= \int_{-2}^{1} \pi y^2 dx - \int_{-2}^{1} \pi y^2 dx$$

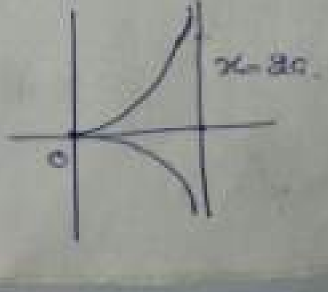
along AB      along $\widehat{ACB}$

$$= \int_{-2}^{1} \pi (-x+3)^2 \, dx - \int_{-2}^{1} \pi (x^2+1)^2 \, dx$$

$$= \frac{117}{5}\pi \text{ cubic unit}$$

④ Find the volume of the solid obtained by the revolution of the cissoid $y^2(2a-x) = x^3$ about its asymptote

$$= 2 \int_{0}^{\infty} \pi (x-2a)^2 dy$$




Change of variables

$x$   0 to 2a

Find the Volume of the Solid obtained by the revolution of the cissoid $y^2(2a-x) = x^3$ about its asymptote


Here curve is revolving about line $x = 2a \parallel$ to $y$ axis. $\therefore$ Volume generated is

$$V = \int_0^{\infty} 2\pi (x-2a)^2 dy$$

= 2 of vol. generated by upper half of the curve

Also $y = \dfrac{x^{3/2}}{\sqrt{2a-x}}$

$$dy = \frac{\sqrt{2a-x}\cdot \frac{3}{2} x^{1/2} - x^{3/2} \cdot \frac{-1}{\sqrt{2a-x}}}{2a-x} \cdot dx$$

$$V = 2\pi \int_0^{2a} (x-2a)^2 \frac{\frac{3}{2}\sqrt{x}(2a-x) + x^{3/2}}{(2a-x)^{3/2}} \cdot dx$$

$$= 2\pi \int_0^{2a} \frac{(2a-x)^2 \frac{\sqrt{x}}{2}\left[6a - 3x + 2x\right]}{(2a-x)^{3/2}} \cdot dx$$

$$2\pi \cdot \frac{1}{2} \int_0^{2a} \sqrt{x}\,\sqrt{2a-x}\,(6a-2x) \cdot dx$$

$$= 2\pi \int_0^{2a} \sqrt{x}\,\sqrt{2a-x}\,(3a-x) \cdot dx$$

either put $x = 2a\sin^2\theta$

or $x = 2a\cos^2\theta$

$$= 16\pi a^3 \int_0^{\pi/2} 3\sin^2\theta\cos^2\theta(3-2\sin^2\theta) \cdot d\theta$$

(Solve by yourself)

$$= 2\pi^2 a^3$$

Volume

$$V_x = \int_{x=a}^{b} \pi y^2 dx$$

$$V_y = \int_{y=c}^{d} \pi x^2 dy$$

$$V_{y=b} = \int_{x=a}^{b} \pi (y-b)^2 \cdot dx$$

$$V_{x=q} = \int_{y=c}^{d} \pi (x-q)^2 dy$$

$$\int_0^{\pi/2} \sin^n\theta \, d\theta \text{ or } \int_0^{\pi/2} \cos^n\theta \, d\theta$$

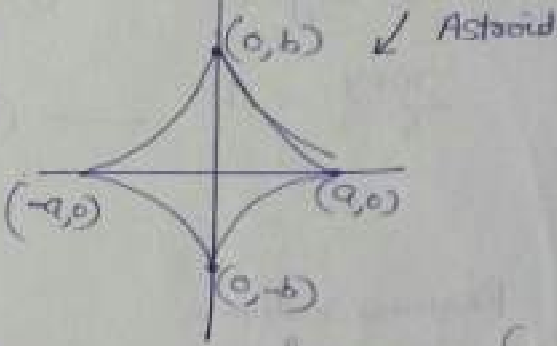
$$= \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots} \left\{ \times \frac{\pi}{2} \text{ if } n \text{ even} \right.$$

$$\int_0^{\pi/2} \sin^n\theta \cos^m\theta \, d\theta$$

$$= \frac{(n-1)(n-3)\cdots(m-1)(m-3)\cdots}{(n+m)(n+m-2)\cdots} \times \frac{\pi}{2} \left\{ \text{if } n \text{ \& } m \text{ both even} \right.$$

Find the Volume of the Solid formed by revolving Astroid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ about $x$-axis

Parametric form of Astroid is




$x = a\cos^3\theta$

$y = b\sin^3\theta$

$$V_x = \int_{-a,\ \theta=0}^{a,\ \theta=\pi} \pi y^2 \cdot \frac{dx}{d\theta} \cdot d\theta$$

$$= \int_0^{\pi} \pi\, b^2 \sin^6\theta \cdot a \cdot 3\cos^2\theta \cdot (\sin\theta) \cdot d\theta$$

$$= \int_0^{\pi} 3\pi a b^2 \sin^7\theta \cdot \cos^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} 3\pi a b^2 \sin^7\theta \cdot \cos^2\theta \, d\theta \quad \left[\text{Being Symmetrical}\right]$$

$$= 2 \cdot \cancel{3}\pi a b^2 \cdot \frac{\overset{2}{\cancel{6}} \cdot 4 \cdot 2 \cdot 1}{\underset{3}{\cancel{9}} \cdot 7 \cdot 5 \cdot \cancel{3}}$$

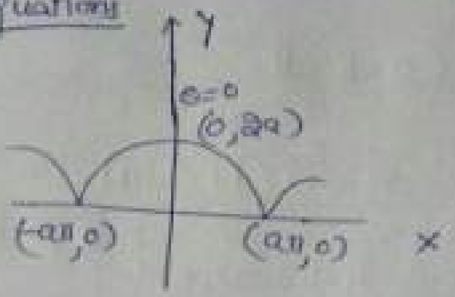
$$\frac{32}{105}\pi a b^2$$

# Parametric equations

Cycloids

(1) $x = a(\theta + \sin\theta)$

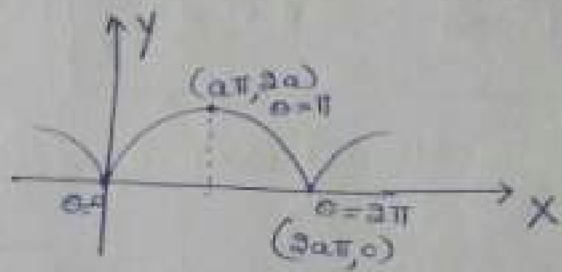
$y = a(1 + \cos\theta)$




$\theta = -\pi$ to $\pi$

(2) $x = a(\theta - \sin\theta)$

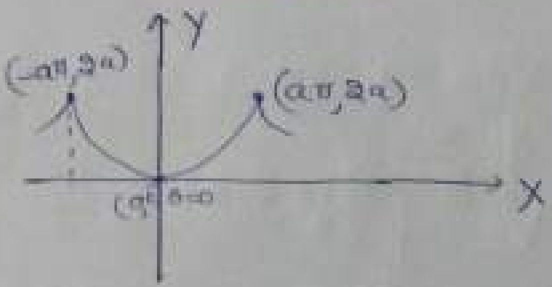
$y = a(1 - \cos\theta)$




$\theta = 0$ to $2\pi$

(3) $x = a(t + \sin t)$

$y = a(1 - \cos t)$



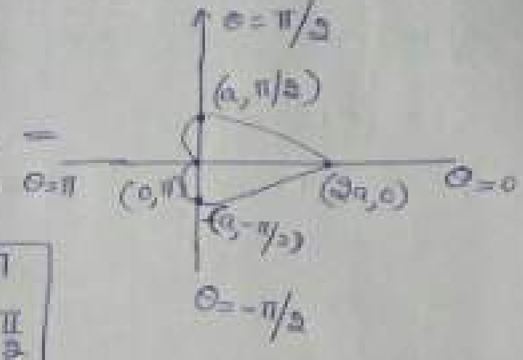

$\theta = -\pi$ to $\pi$

Revolution about x-axis / about its base / about the tangent at the vertex

$$V_x = \int_{\theta=\alpha}^{\beta} \pi y^2 \frac{dx}{d\theta} \cdot d\theta$$

# Cardioids

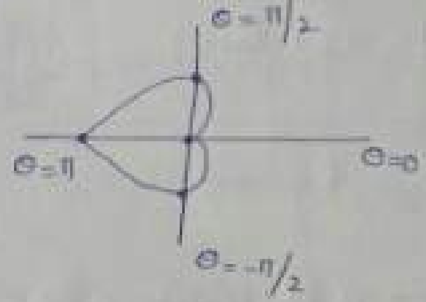
$r = a(1+\cos\theta)$



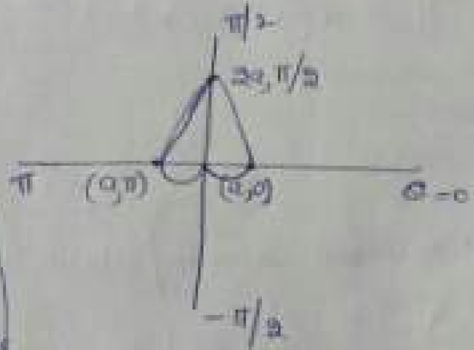

Revolution about Initial axis $\theta \to 0$ to $\pi$

Revolution about $\perp$ axis $\theta \to -\frac{\pi}{2}$ to $\frac{\pi}{2}$

$r = a(1-\cos\theta)$




$r = a(1+\sin\theta)$



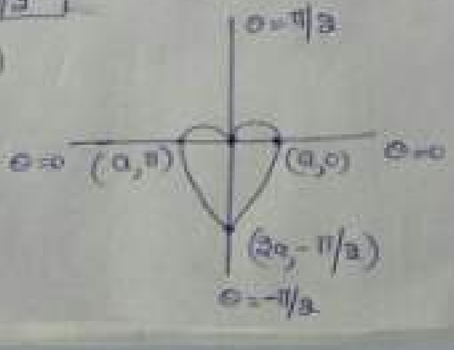

Initial axis rotation

$\theta = 0$ to $\pi$

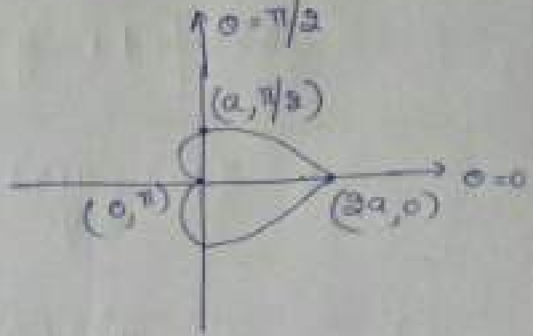
axis perpendicular to initial axis

$\theta = -\pi/2$ to $\pi/2$

$r = a(1-\sin\theta)$



find the Volume of the Solid formed by revolving $r = a(1+\cos\theta)$ about initial axis.




<u>Initial axis</u>

$$V_{\theta=0} = \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

$$= \int_0^{\pi} -\frac{2}{3}\pi a^3 (1+\cos\theta)^3 \cdot -\sin\theta \, d\theta$$

$$= -\frac{2}{3}\pi a^3 \left. \frac{(1+\cos\theta)^4}{4} \right]_0^{\pi}$$

$$= -\frac{2}{3}\pi a^3 \cdot -\frac{16}{4}$$

$$= \frac{8}{3}\pi a^3$$

Axis perpendicular to Initial axis

$$V = \int_{-\pi/2}^{\pi/2} \frac{2}{3}\pi a^3 (1+\cos\theta) \cdot \cos\theta \, d\theta$$

[even function]

$$= 2\int_0^{\pi/2} \frac{2}{3}\pi a^3 \left(\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{4}{3}\pi a^3 \int_0^{\pi/2} \left(\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{4}{3}\pi a^3 \left[\sin\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

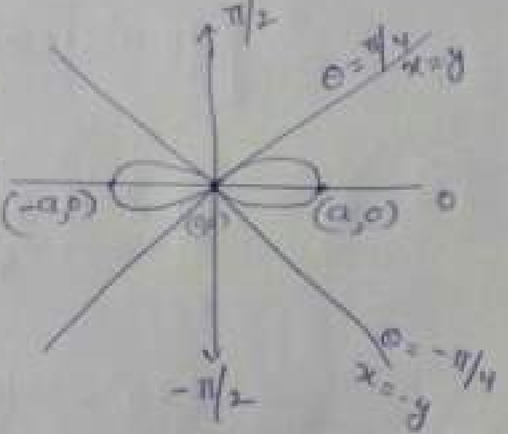
$$= \frac{4}{3}\pi a^3 \left[1 + \frac{\pi}{4} + 0\right]$$

$$= \frac{4}{3}\pi a^3 \frac{(\pi+4)}{4}$$

$$\boxed{V_{\theta=\pi/2} = \pi^2 a^3 \left(1+\frac{4}{\pi}\right)}$$

<u>Lemniscate</u> $r^2 = a^2 \cos 2\theta$

Volume of revolution about y-axis




$$V = \int_{-\pi/4}^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi a^3 (1-2\sin^2\theta)^{3/2} \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{4}{3}\pi a^3 (1-\sin^2\phi)^{3/2} \frac{1}{\sqrt{2}}\cos\phi \, d\phi$$

$$= \frac{4}{3}\pi a^3 \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4\phi \, d\phi = \frac{4}{3}\pi a^3 \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}$$

$$\boxed{V = \frac{\pi^2 a^3}{4\sqrt{2}}}$$

put

$\sqrt{2}\sin\theta = \sin\phi$

$\sqrt{2}\cos\theta \, d\theta = \cos\phi \, d\phi$

when $\theta = 0$
$\Rightarrow \phi = 0$

when $\theta = \frac{\pi}{4}$
$\Rightarrow \phi = \pi/2$

**Example 12.** *The smaller segment of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *cut off by the chord*

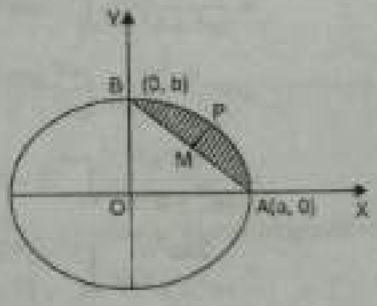
Revision

$\frac{x}{a}+\frac{y}{b}=1$ *revolves completely about the chord, show that the volume generated is*

$$\frac{1}{6}\pi(10-3\pi)a^2b^2(a^2+b^2)^{-1/2}.$$

**Sol.** Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of the chord is $\frac{x}{a}+\frac{y}{b}=1.$




This chord passes through A($a$, 0) and B (0, $b$) of the ellipse. The shaded region revolves about line AB (any line i.e., neither ∥ to $x$-axis nor ∥ to $y$-axis)

∴ Take A as the fixed point on the line AB

∴ Volume of the solid generated by region

$$=\pi\int PM^2\, d(AM)$$

Let P($a\cos\theta$, $b\sin\theta$) be any point on the ellipse then PM = Length of perpendicular distance from ($a\cos\theta$, $b\sin\theta$) on $\frac{x}{a}+\frac{y}{b}=1$

$$=\frac{\frac{a\cos\theta}{a}+\frac{b\sin\theta}{b}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{\cos\theta+\sin\theta-1}{\sqrt{a^2+b^2}}ab$$

$$AM^2=AP^2-PM^2=(a\cos\theta-a)^2+(b\sin\theta)^2-\left(\frac{\cos\theta+\sin\theta-1}{\sqrt{a^2+b^2}}ab\right)^2$$

$$=a^2(\cos\theta-1)^2+b^2\sin^2\theta-\frac{a^2b^2}{a^2+b^2}((\cos\theta-1)+\sin\theta)^2$$

$$=a^2(\cos\theta-1)^2+b^2\sin^2\theta-\frac{a^2b^2}{a^2+b^2}(\cos\theta-1)^2$$

$$-\frac{a^2b^2}{a^2+b^2}\sin^2\theta-\frac{2a^2b^2}{a^2+b^2}\sin\theta(\cos\theta-1)$$

$$= \frac{a^4}{a^2+b^2}(\cos\theta-1)^2 + \frac{b^4}{a^2+b^2}\sin^2\theta - \frac{2a^2b^2}{a^2+b^2}\sin\theta(\cos\theta-1)$$

$$= \left(\frac{a^2(\cos\theta-1)-b^2\sin\theta}{\sqrt{a^2+b^2}}\right)^2$$

$$\therefore \quad \text{AM} = \pm\frac{a^2(\cos\theta-1)-b^2\sin\theta}{\sqrt{a^2+b^2}} = \frac{a^2(1-\cos\theta)+b^2\sin\theta}{\sqrt{a^2+b^2}}$$

$\therefore$ Required volume

$$= \pi\int_{\theta=0}^{\pi/2}\frac{(\cos\theta+\sin\theta-1)^2\,a^2b^2d}{a^2+b^2}\left[\frac{a^2(1-\cos\theta)+b^2\sin\theta}{\sqrt{a^2+b^2}}\right]d\theta$$

$$= \frac{\pi a^2b^2}{(a^2+b^2)^{3/2}}\int_0^{\pi/2}(\cos\theta+\sin\theta-1)^2\,(a^2\sin\theta+b^2\cos\theta)\,d\theta$$

$$= \frac{\pi a^2b^2}{(a^2+b^2)^{3/2}}\int_0^{\pi/2}(\cos^2\theta+\sin^2\theta+1+2\sin\theta\cos\theta-2\sin\theta-2\cos\theta)\,(a^2\sin\theta+b^2\cos\theta)\,d\theta$$

$$= \frac{\pi a^2b^2}{(a^2+b^2)^{3/2}}\int_0^{\pi/2}(2+2\sin\theta\cos\theta-2\sin\theta-2\cos\theta)\,(a^2\sin\theta+b^2\cos\theta)\,d\theta$$

$$= \frac{2\pi a^2b^2}{(a^2+b^2)^{3/2}}\int_0^{\pi/2}(a^2\sin\theta+a^2\sin^2\theta\cos\theta-a^2\sin^2\theta-a^2\sin\theta\cos\theta + b^2\cos\theta+b^2\sin\theta\cos^2\theta-b^2\sin\theta\cos\theta-b^2\cos^2\theta)\,d\theta$$

$$= \frac{2\pi a^2b^2}{(a^2+b^2)^{3/2}}\left\{a^2.1+a^2\frac{1}{3.1}-a^2\frac{1}{2}\frac{\pi}{2}-(a^2+b^2)\frac{1}{2}+b^2+b^2\frac{1}{3}-b^2\frac{1}{2}\frac{\pi}{2}\right\}$$

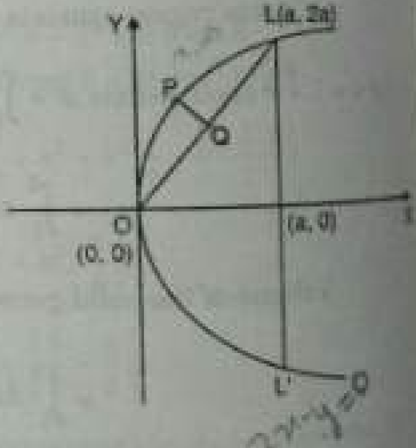
$$= \frac{2\pi a^2b^2}{(a^2+b^2)^{3/2}}\left[a^2\left(1+\frac{1}{3}-\frac{1}{2}\right)+b^2\left(-\frac{1}{2}+1+\frac{1}{3}\right)-(a^2+b^2)\frac{\pi}{4}\right]$$

$$= \frac{2\pi a^2b^2}{(a^2+b^2)^{3/2}}\left[\frac{5}{6}(a^2+b^2)-\frac{\pi}{4}(a^2+b^2)\right] = \frac{\pi a^2b^2}{\sqrt{a^2+b^2}}\left[\frac{5}{3}-\frac{\pi}{2}\right].$$

**Example 13.** Find t...

15 15 15

**Example 16.** *The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Show that the volume of the solid so formed is $\frac{2\sqrt{5}}{75}\pi a^3$.* (P.T.U. May 2001)




**Sol.** Equation of the parabola is

$$y^2 = 4ax \qquad ...(1)$$

Let LL′ be the latus rectum of the parabola.

We know that length of the latus rectum is $4a$.

$\therefore$ Coordinates of L are $(a, 2a)$ equation of OL is

$$y - 0 = \frac{2a-0}{a-0}(x-0) \quad \text{or} \quad y = 2x$$

Let $P(x, y)$ be any point on the parabola then $PQ = \perp$ distance from P on OL

$$= \frac{|2x-y|}{\sqrt{4+1}} = \frac{|2x-y|}{\sqrt{5}}$$

$$OQ^2 = OP^2 - PQ^2 = (x^2+y^2) - \frac{(2x-y)^2}{5} = \frac{5x^2+5y^2-4x^2-y^2+4xy}{5}$$

$$\therefore \quad OQ^2 = \frac{x^2+4y^2+4xy}{5} = \frac{(x+2y)^2}{5}$$

$$\therefore \quad OQ = \frac{x+2y}{\sqrt{5}}$$

When we rotate the curve about OL then volume of the solid

$$= \int \pi(PQ)^2 \, d(OQ) = \pi\int \frac{(2x-y)^2}{5} \, d\left(\frac{x+2y}{\sqrt{5}}\right)$$

Replace y by $2\sqrt{ax}$ from (1)

$$= \pi \int \frac{(2x - 2\sqrt{ax})^2}{5}\, d\left[\frac{x + 4\sqrt{ax}}{\sqrt{5}}\right]$$

$$= \pi \int_0^a \frac{4}{5\sqrt{5}} (x - \sqrt{ax})^2 \left(1 + \frac{4\sqrt{9}}{2\sqrt{x}}\right) dx$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a (x^2 - 2\sqrt{a}\, x^{3/2} + ax) \left(\frac{\sqrt{x} + 2\sqrt{a}}{\sqrt{x}}\right) dx$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a (x^{3/2} - 2\sqrt{a}x + ax^{1/2})\,(x^{1/2} + 2\sqrt{a})\, dx$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a [x^2 - 2\sqrt{a}\, x^{3/2} + ax + 2\sqrt{a}\, x^{3/2} - 4ax + 2a\sqrt{a}\, x^{1/2}]\, dx$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a [x^2 - 3ax + 2a\sqrt{a}\, x^{1/2}]\, dx$$

$$= \frac{4\pi}{5\sqrt{5}} \left[\frac{x^3}{3} - 3a\frac{x^2}{2} + 2a^{3/2} \cdot \frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{4\pi}{5\sqrt{5}} \left[\frac{a^3}{3} - \frac{3a^3}{2} + \frac{4}{3}a^3\right]$$

$$= \frac{4\pi}{5\sqrt{5}} \frac{2 - 9 + 8}{6} a^3 = \frac{2\pi a^3}{15\sqrt{5}} = \frac{2\sqrt{5}}{75} \pi a^3$$

Hence volume of the solid $= \frac{2\sqrt{5}}{75} \pi a^3$.