Roll No.

Total No. of Pages : 02

Total No. of Questions : 09

B.Tech. (AI & DS/ AI & ML/ Block Chain / CSE / CE / CS / Computer Science and Design / EEE / EE / ECE / Electronics & Telecommunication Engineering / FT / IT / ME / Robotics & Artificial Intelligence/ Internet of Things and Cyber Security including Block Chain Technology) (Sem.– 1)

ENGINEERING MATHEMATICS-I

Subject Code : BTAM101-23

M.Code : 93796

Date of Examination: 08-12-2023

Time : 3 Hrs. **Max. Marks : 60**

INSTRUCTIONS TO CANDIDATES :

1. SECTION-A is COMPULSORY consisting of TEN questions carrying TWO marks each.
2. SECTION - B & C have FOUR questions each.
3. Attempt any FIVE questions from SECTION B & C carrying EIGHT marks each.
4. Select atleast TWO questions from SECTION - B & C.

SECTION-A

1. Answer briefly :

a) What do you mean by bounded and unbounded sequences?

b) Prove that the sequence $\frac{2n-7}{3n+2}$ is monotonically increasing.

c) Define p-Test for the series.

d) Find the length of the arc of the parabola $x^2 = 4ay$ extending from the vertex to one extremity of the latus rectum.

e) Test for convergence of integral $\int_0^\infty \frac{\sin^2 x}{x^2} dx$.

f) Define Beta function.

g) State Euler's theorem for homogeneous function.

h) Show that the function $f(x, y) = \frac{2x^2 y}{x^4 + y^2}$ has no limit as $(x, y) \to (0, 0)$.

i) Evaluate $\int\limits_{1}^{\log 8}\int\limits_{0}^{\log y} e^{x+y}\,dx\,dy$.

j) Evaluate $\int\limits_{0}^{1}\int\limits_{0}^{1-x}\int\limits_{0}^{2-x} x\,y\,z\,dz\,dy\,dx$.

SECTION-B

2. Prove that the sequence a_n where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \cdots\cdots \frac{1}{2n}$ is convergent.

3. Discuss the convergence or divergence of the series $\frac{1}{1.2.3} + \frac{2}{2.3.4} + \frac{5}{3.4.5} + \cdots\cdots$

4. Find the surface of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis.

5. Prove that $\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$ where $m > 0, n > 0$.

SECTION-C

6. If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$.

7. Show that the first four terms of the Maclaurin's expansion of $e^{ax}\cos by$ are :

$$1 + ax + \frac{1}{\underline{|2}}(a^2x^2 - b^2y^2) + \frac{1}{\underline{|3}}(a^3x^3 - 3ab^2xy^2).$$

8. Evaluate $\int\limits_{0}^{\infty}\int\limits_{0}^{x} xe^{-x^2/y}\,dy\,dx$ by change of order of integration.

9. Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ over the region $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$

NOTE : Disclosure of Identity by writing Mobile No. or Marking of passing request on any paper of Answer Sheet will lead to UMC against the Student.

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B. Tech (Semester-1) (Dec 2023)

ENGG. MATHEMATICS-I

Subject Code: BTAM-101-23

M. Code: 93796

B.Tech (AI&DS/AI&ML/CSE/EE/ECE/CSE-DS/IT)

SECTION-A

Q1 Answer briefly:-

(a) What do you mean by bounded and unbounded sequence?

→ A sequence $\{a_n\}$ is said to be bounded above if there exist a real no. 'k' such that $a_n \le k \ \forall \ n \in \mathbb{N}$, k is called an upper bound of $\{a_n\}$.

A sequence $\{a_n\}$ is said to be bounded below if there exist a real no. 'h' such that $h \le a_n \ \forall \ n \in \mathbb{N}$, h is called an lower bound of $\{a_n\}$.

A sequence $\{a_n\}$ is said to be bounded if it is bounded above as well as bounded below i.e. if ∃ two real no. 'h' and 'k' such that:-

$$h \le a_n \le k \quad , n \in \mathbb{N}$$

A sequence is said to be unbounded if it is not bounded. A sequence is unbounded if given $\Delta > 0$ however large ∃ $m \in \mathbb{N}$ such that $|a_n| > \Delta \ \forall \ n \ge m$.

(b) Prove that the sequence $\frac{2n-7}{3n+2}$ is monotonically increasing.

→ A sequence $\{a_n\}$ is said to be monotonically increasing if $a_n \le a_{n+1}$.

Here, $a_n = \frac{2n-7}{3n+2}$

$$a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2}$$

$$= \frac{2n+2-7}{3n+3+2}$$

$$= \frac{2n-5}{3n+5}$$

$$a_{n+1} - a_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$$

$$= \frac{(2n-5)(3n+2) - (2n-7)(3n+5)}{(3n+5)(3n+2)}$$

$$= \frac{6n^2+4n-15n-10 - [6n^2+10n-21n-35]}{(3n+5)(3n+2)}$$

$$= \frac{\cancel{6n^2} - \cancel{11n} - 10 - \cancel{6n^2} + \cancel{11n} + 35}{(3n+5)(3n+2)}$$

$$= \frac{25}{(3n+5)(3n+2)} > 0$$

$$= a_{n+1} - a_n > 0$$

$$= a_{n+1} > a_n$$

$\therefore$ $\{a_n\}$ is Monotonically Increasing sequence.

<u>Hence Proved</u>

(c) Define p-Test for the Series:-

$$\Sigma a_n = \sum_{n=1}^{\infty} \frac{1}{n^p} > 0, \text{ cgs if } p > 1$$

$$\text{dgs if } p < 1$$

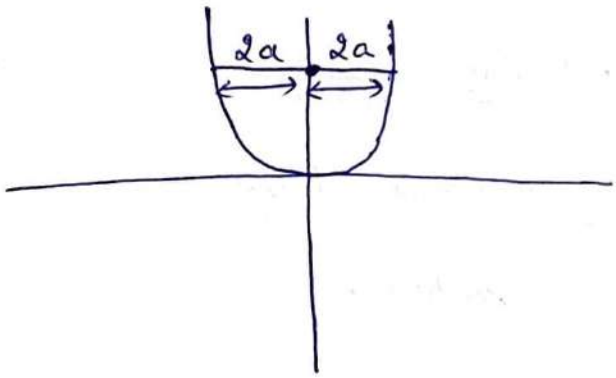
for eg:- $\Sigma \frac{1}{n^2}$, here $p = 2 > 1$

$\therefore$ Σa_n cgs

$\Sigma \frac{1}{n^{1/2}}$ = Here, $p = \frac{1}{2} < 1$

$\therefore$ Σa_n dgs

(d) Find the length of the arc of the parabola $x^2 = 4ay$ extending from the vertex to one extremity of the latus rectum.



Given, $x^2 = 4ay$

$\longrightarrow y = \frac{x^2}{4a}$

$\longrightarrow \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

Length of the arc $= \int_0^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^{2a} \sqrt{1+\left(\frac{x}{2a}\right)^2}\, dx$

$= \frac{1}{2a} \int_0^{2a} \sqrt{4a^2+x^2}\, dx$

By using formula, $\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \log \left| x+\sqrt{a^2+x^2} \right|$

$= \frac{1}{2a}\left[\frac{x}{2} \sqrt{(2a)^2+x^2} + \frac{(2a)^2}{2} \log \left| x+\sqrt{(2a)^2+x^2} \right| \right]_0^{2a}$

$= \frac{1}{2a}\left[\frac{2a}{2} \sqrt{(2a)^2+(2a)^2} + \frac{4a^2}{2} \log \left| 2a+\sqrt{(2a)^2+(2a)^2} \right| \right] - \frac{1}{2a}\left[0 + \frac{(2a)^2}{2} \log \right|$

$0+\sqrt{(2a)^2+(0)^2} \Big|$

$= \frac{1}{2a}\left[a\sqrt{4a^2+4a^2} + 2a^2 \log\left|2a+\sqrt{4a^2+4a^2}\right|\right] - \frac{1}{2a}\left[\frac{4a^2}{2}\log|2a$

$= \frac{1}{2a}\left[a\sqrt{8a^2} + 2a^2\log\left|2a+\sqrt{8a^2}\right|\right] - \frac{1}{2a}\left[2a^2\log 2a\right]$

$= \frac{1}{2a}\left[2\sqrt{2}a^2 + 2a^2\log\left|2a+2\sqrt{2}a\right| - \frac{1}{2a}\left[2a^2\log 2a\right]\right]$

$= a\left[\sqrt{2} + \log\frac{|2a+2\sqrt{2}a|}{2a}\right]$

$= a\left[\sqrt{2} + \log(1+\sqrt{2})\right] \longrightarrow$ Ans

(e) Test for convergence of integral $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

$\rightarrow$ $\sin^2 x$ will always be positive.

By p-test:→ $2 > 1$

$\therefore$ Integral $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ cgs.

(f) Define Beta function.

The Integral $I = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ where, $m>0$ and $n>0$ is called a beta function and is denoted by $\beta(m,n)$.

Eg:- $\int_0^1 x^3(1-x)^5\,dx$

Here, $m-1=3$ $\quad$ $n-1=5$

$\boxed{m=4}$ $\quad$ $\boxed{n=6}$

$\beta(m,n) = \beta(4,6)$

Eg 2:- $\int_0^1 x^{1/3}(1-x)^{-2}\,dx$

Here, $m-1 = \frac{1}{3}$ $\qquad$ $n-1 = -2$

$m = \frac{1}{3}+1$ $\qquad$ $n = -2+1$

$\boxed{m = \frac{4}{3}}$ $\qquad$ $\boxed{n = -1}$

As $n < 0$, due to which condition of Beta function is not satisfied. $\therefore$ It is not a beta function.

(g) State Euler's theoram for homogenous function.

If z is homogenous function of x and y of order n

then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \forall\ x, y \in$ Domain of the function.

For eg:- $z = \frac{x^3y^3}{x^3+y^3}$

$\rightarrow z = \frac{\cancel{x^3}\left(\frac{y}{x}\right)^3 \times x^3}{\cancel{x^3}\left(1+\left(\frac{y}{x}\right)^3\right)}$

Here, z is a homogenous function of degree 3.

By using Euler theoram:-

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$,

where $\boxed{n=3}$

(h) Show that the function $f(x,y) = \frac{2x^2y}{x^4+y^2}$ has no limit as $(x,y) \to (0,0)$.

$\rightarrow \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{h\to0} f\frac{(a+h,b)-f(a,b)}{h}$

$= \lim_{h\to0} f\frac{(0+h,0)-f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2(h)^2\times 0}{(h)^4+(0)^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$$= 0$$

$$\left(\frac{\partial f}{\partial y}\right)_{0,0} = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{2(0)^2\times k}{(0)^4+k^2} - 0}{k}$$

$$= \lim_{k\to 0} \frac{0}{k}$$

$$= 0$$

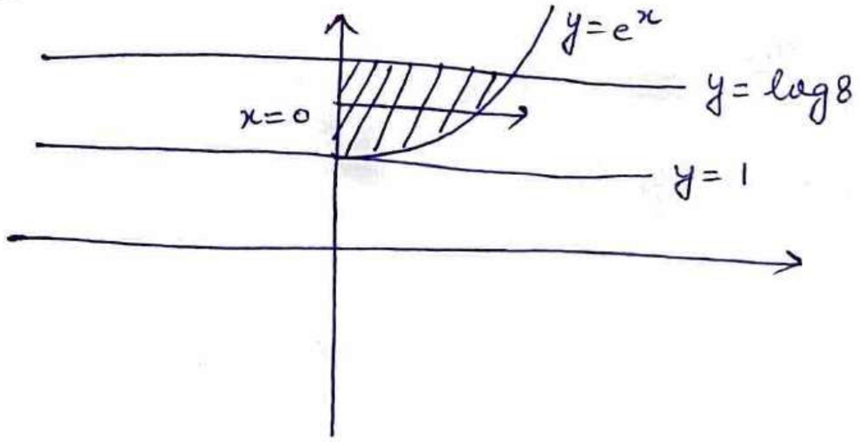
Hence, $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ has no limit as $(x,y) \longrightarrow (0,0)$.

(i) Evaluate $\displaystyle\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy$

→ Region of Integration :-

$x=0$ to $x=\log_e y$

$y=1$ to $y=\log_e 8$



$$\rightarrow \int_1^{\log 8}\left[\int_0^{\log y} e^{x+y}\,dx\right]dy$$

$$\rightarrow \int_1^{\log 8}\left[e^y\,[e^x]_0^{\log y}\right]dy$$

$$\rightarrow \int_1^{\log 8} e^y\,[e^{\log y}-e^0]\,dy$$

$$\rightarrow \int_1^{\log 8} \underset{II}{e^y}\,\underset{I}{[y-1]}\,dy$$

By using ILATE

$$\rightarrow \left[(y-1)e^y\right]_1^{\log 8} - \int_1^{\log 8} 1\cdot e^y\,dy$$

$$\rightarrow \left[(y-1)e^y\right]_1^{\log 8} - \left[e^y\right]_1^{\log 8}$$

$$\rightarrow (\log 8-1)e^{\log 8} - [e^{\log 8}-e^1]$$

$$\rightarrow (\log 8-1)e^{\log 8} - [e^{\log 8}-e]$$

$$\rightarrow (\log 8-1)e^{\log 8} - e^{\log 8}+e$$

$$\rightarrow e^{\log 8}[\log 8-1-1]+e$$

$$\rightarrow e^{\log 8}[\log 8-2]+e$$

$$\rightarrow 8\,[\log 8-2]+e$$

$\longrightarrow 8\log 8 - 16 + e$ <u>Ans</u>

(j) Evaluate :- $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz \, dz \, dy \, dx$

$$\longrightarrow \int_0^1 \int_0^{1-x} \left[\frac{xyz^2}{2}\right]_0^{2-x} dy \, dx$$

$$\longrightarrow \int_0^1 \int_0^{1-x} \left[\frac{xy(2-x)^2}{2} dy\right] dx$$

$$\longrightarrow \int_0^1 \left[\frac{x(2-x)^2}{2} \cdot \frac{y^2}{2}\right]_0^{1-x} dx$$

$$\longrightarrow \int_0^1 x \frac{(2-x)^2}{2} \frac{(1-x)^2}{2} dx$$

$$\longrightarrow \frac{1}{4} \int_0^1 x(2-x)^2 (1-x)^2 dx$$

$$\longrightarrow \frac{1}{4} \int_0^1 x(4+x^2-4x)(1+x^2-2x) dx$$

$$\longrightarrow \frac{1}{4} \int_0^1 (4x+x^3-4x^2)(1+x^2-2x) dx$$

$$\longrightarrow \frac{1}{4} \int_0^1 4x+4x^3-8x^2+x^3+x^5-2x^4-4x^2-4x^4+8x^3 \, dx$$

$$\longrightarrow \frac{1}{4} \int_0^1 4x+13x^3-12x^2-6x^4+x^5 \, dx$$

$$\longrightarrow \frac{1}{4} \left[\frac{4x^2}{2} + \frac{13x^4}{4} - \frac{12x^3}{3} - \frac{6x^5}{5} + \frac{x^6}{6}\right]_0^1$$

$$\longrightarrow \frac{1}{4} \left[2 + \frac{13}{4} - \frac{12}{3} - \frac{6}{5} + \frac{1}{6}\right]$$

$\rightarrow \frac{1}{4}\left[-2+\frac{13}{4}-\frac{6}{5}+\frac{1}{6}\right]$

$\rightarrow \frac{1}{4}\left[-2+\frac{65-24}{20}+\frac{1}{6}\right]$

$\rightarrow \frac{1}{4}\left[-2+\frac{41}{20}+\frac{1}{6}\right]$

$\rightarrow \frac{1}{4}\left[\frac{-120+123+10}{60}\right]$

$\rightarrow \frac{1}{4}\left[\frac{13}{60}\right]$

$\rightarrow \frac{13}{240}$ Ans

Section- B

Q2 Prove that the sequence where $a_n = \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}$ is convergent.

$\longrightarrow \quad a_n = \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}$

$\longrightarrow \quad a_{n+1} = \frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}+\cdots+\frac{1}{2n}+\frac{1}{2n+1}+\frac{1}{2n+2}$

$$\longrightarrow \quad a_{n+1}-a_n = \frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}$$

$$= \frac{1}{2n+1}-\left[\frac{1}{2(n+1)}-\frac{1}{(n+1)}\right]$$

$$= \frac{1}{2n+1}-\frac{1}{2(n+1)}$$

$$= \frac{\cancel{2n}+2-\cancel{2n}-1}{(2n+1)(2n+2)}$$

$$= \frac{1}{(2n+1)(2n+2)} > 0$$

Since, $a_{n+1} > a_n$

$\therefore$ The given sequence is monotonically Increasing

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} < \frac{1}{n} + \frac{1}{n} + \cdots n \text{ terms}$$

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} < \frac{1}{\cancel{n}}(\cancel{n})$$

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} < 1$$

for $\{a_n\}$ upper bound Exists

Hence, sequence $\{a_n\}$ is bounded above

So, monotonically Increasing + Bounded above

$\therefore$ Convergent

<u>Hence Proved</u>

<u>Q3</u> Discuss the convergence or divergence of the series

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{2\cdot3\cdot4} + \cdots$$

$\rightarrow$ Here, $a_n = \dfrac{2n-1}{n(n+1)(n+2)}$

Let $b_n = \dfrac{1}{n^2}$

So, $\dfrac{a_n}{b_n} = \dfrac{(2n-1)}{n(n+1)(n+2)} \times n^2$

$$= \frac{a_n}{b_n} = \frac{n^3\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{a_n}{b_n} = 2$$

hich is finite and non-zero.

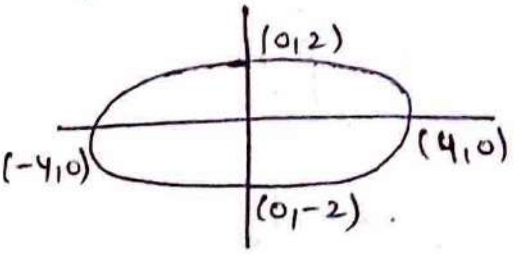
$\therefore$ By comparision test both Σa_n and Σb_n converge or diverge together.

But by p-test Σb_n cgs as $2 > 1$

$\therefore$ By comparision test

Σa_n converges

Q4 Find the surface of the solid generated by the revolution of the ellipse $x^2+4y^2=16$ about its major axis.



$\longrightarrow$ $x^2+4y^2=16$ (Given)

$\longrightarrow$ $\frac{x^2}{16}+\frac{4y^2}{16}=1$

$\longrightarrow$ $y^2=\frac{16-x^2}{4}$ or $4x^2+16y^2=64$

Thus $\boxed{16y^2=64-4x^2}$

As $x^2+4y^2=16$

$\rightarrow 2x+8y\frac{dy}{dx}=0$ $\therefore \left[\frac{dy}{dx}=\frac{-x}{4y}\right]$

Thus, $\left(\frac{dy}{dx}\right)^2=\frac{x^2}{16y^2}$ Now, $1+\left(\frac{dy}{dx}\right)^2=\frac{16y^2+x^2}{16y^2}$

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{ds}{dx}=\sqrt{\frac{16y^2+x^2}{16y^2}}$$

$$\frac{ds}{dx}=\frac{\sqrt{16y^2+x^2}}{\sqrt{(4)^2y^2}}$$

$$\frac{ds}{dx} = \frac{\sqrt{64-4x^2+x^2}}{64-4x^2}$$

$$\frac{ds}{dx} = \frac{\sqrt{64-3x^2}}{\sqrt{16y^2}}$$

$$\frac{ds}{dx} = \frac{\sqrt{64-3x^2}}{4y} \quad - (1)$$

$$S = 2\pi \int_{-4}^{4} y\, ds \quad = 2\pi \int_{-4}^{4} \cancel{y} \frac{\sqrt{64-3x^2}}{4\cancel{y}}$$

$$S = \frac{2\pi}{4} \int_{-4}^{4} \sqrt{64-3x^2}\, dx$$

$$S = \frac{2\pi}{4} \cdot 2 \cdot \int_{0}^{4} \sqrt{64-3x^2}\, dx$$

$$S = \pi \cdot \sqrt{3} \int_{0}^{4} \sqrt{\frac{64}{3} - x^2}\, dx$$

$$S = \pi \cdot \sqrt{3} \left[\frac{x\sqrt{\frac{64}{3} - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{\sqrt{64/3}} \right]$$

$$S = \pi \cdot \sqrt{3} \left[\frac{4\sqrt{\frac{64}{3} - 16}}{2} + \frac{64}{3 \cdot 2} \sin^{-1} \frac{4\sqrt{3}}{6} - 0 \right]$$

$$S = \sqrt{3}\pi \left[2\sqrt{\frac{16}{3}} + \frac{32}{3} \cdot \frac{\pi}{3} \right]$$

$$S = \sqrt{3}\pi \left[\frac{2 \times 4}{\sqrt{3}} + \frac{32\pi}{9} \right]$$

$$S = \sqrt{3}\pi \left[\frac{8}{\sqrt{3}} + \frac{32\pi}{9} \right]$$

$$\boxed{S = 8\pi + \frac{32\sqrt{3}\pi^2}{9}} \longrightarrow \text{Answer}$$

Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where $m>0, n>0$.

→ Proof :- $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

Put $x = tz$

$dx = t\,dz$

When $x=0$, $z=0$

$x=\infty$, $z=\infty$

$$\Gamma(n) = \int_0^\infty (tz)^{n-1} e^{-tz} t\,dz$$

$$= \int_0^\infty t^n z^{n-1} e^{-tz} dz$$

Multiply both sides by $e^{-t} t^{m-1}$

$$\rightarrow \Gamma(n) e^{-t} t^{m-1} = e^{-t} t^{m-1} \int_0^\infty t^n z^{n-1} e^{-tz} dz$$

$$\rightarrow \Gamma(n) e^{-t} t^{m-1} = \int_0^\infty t^{m+n-1} z^{n-1} e^{-t(1+z)} dz$$

Integrating both sides from 0 to ∞ w.r.t. 't' on both sides:-

$$\rightarrow \Gamma(n) \int_0^\infty e^{-t} t^{m-1} dt = \int_0^\infty z^{n-1} \left[\int_0^\infty t^{m+n-1} e^{-t(z+1)} dt \right] dz$$

$$\rightarrow \Gamma(n)\Gamma(m) = \int_0^\infty z^{n-1} \left[\int_0^\infty t^{m+n-1} e^{-t(z+1)} dt \right] dz$$

Put $t(z+1) = y$

$\rightarrow t = \dfrac{y}{z+1}$

$$dt = \frac{dy}{z+1}$$

when $t=0$, $y=0$

$t=\infty$, $y=\infty$

$$\longrightarrow \Gamma(n)\,\Gamma(m) = \int_0^\infty z^{n-1}\left[\int_0^\infty \left(\frac{y}{z+1}\right)^{m+n-1} e^{-y}\cdot\frac{dy}{z+1}\right] dz$$

$$\longrightarrow \Gamma(n)\,\Gamma(m) = \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}}\left[\int_0^\infty y^{m+n-1} e^{-y}\, dy\right] dz$$

$$\longrightarrow \Gamma(n)\,\Gamma(m) = \Gamma(m+n)\int_0^\infty \frac{z^{n-1}\,dz}{(1+z)^{m+n}}$$

$$\longrightarrow \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(m+n)} = \beta(m,n)$$ [By property 2 of Beta function]

Section - C

Q6 If $V = r^m$ where $r^2 = x^2+y^2+z^2$, show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$ $= m(m+1)\, r^{m-2}$.

$$\longrightarrow r = (x^2+y^2+z^2)^{1/2}$$

$$\longrightarrow r^m = (x^2+y^2+z^2)^{m/2}$$

$$\longrightarrow V = (x^2+y^2+z^2)^{m/2}$$

$$\longrightarrow \frac{\partial V}{\partial x} = \frac{m}{\cancel{2}}(x^2+y^2+z^2)^{\frac{m-2}{2}} \times \cancel{2}x$$

$$\longrightarrow \frac{\partial V}{\partial x} = mx\,(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$\longrightarrow \frac{\partial^2 V}{\partial x^2} = m\left[x\cdot\frac{m-2}{\cancel{2}}(x^2+y^2+z^2)^{\frac{m-4}{2}} \times \cancel{2}x\right] + \left[(x^2+y^2+z^2)^{\frac{m-2}{2}}\cdot 1\right]$$

$$\longrightarrow \frac{\partial^2 V}{\partial x^2} = m\left[x^2(m-2)(x^2+y^2+z^2)^{\frac{m-4}{2}}\right] + \left[(x^2+y^2+z^2)^{\frac{m-2}{2}}\right]$$

$$\frac{\partial^2 V}{\partial x^2} = m(m-2)x^2(x^2+y^2+z^2)^{\frac{m-4}{2}} + m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$\frac{\partial^2 V}{\partial y^2} = m(m-2)y^2(x^2+y^2+z^2)^{\frac{m-4}{2}} + m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$\frac{\partial^2 V}{\partial z^2} = m(m-2)z^2(x^2+y^2+z^2)^{\frac{m-4}{2}} + m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m-2)(x^2+y^2+z^2)^{\frac{m-4}{2}}(x^2+y^2+z^2) + 3m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$= m(m-2)r^{m-4}r^2 + 3m(r)^{m-2}$$

$$= m(m-2)r^{m-2} + 3m\,r^{m-2}$$

$$= [m(m-2)+3m]\,r^{m-2}$$

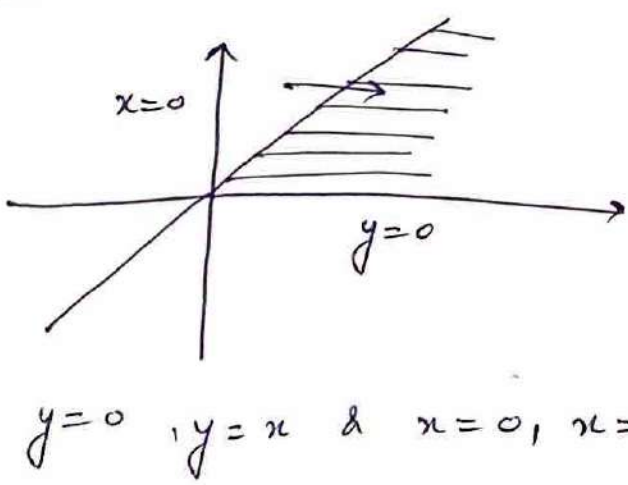
$$= m[m-2+3]\,r^{m-2}$$

$$= m[m+1]\,r^{m-2} = \text{R.H.S}$$

Hence verified

Q8 Evaluate $\int_0^\infty \int_0^x x e^{-x^2/y}\,dy\,dx$ by change of order of integration.

→



$y=0$, $y=x$ & $x=0$, $x=\infty$

Changing the order:-

→ $\int_0^\infty \int_y^\infty x e^{-x^2/y}\,dx\,dy$

Put $x^2=t$

$$\rightarrow 2x\,dx = dt$$

$$\rightarrow x\,dx = \frac{dt}{2}$$

$$\rightarrow \frac{1}{2}\int_0^\infty \left[\int_{y^2}^\infty e^{-t/y}\,dt\right] dy$$

$$\rightarrow \frac{1}{2}\int_0^\infty \left[\frac{e^{-t/y}}{-1/y}\right]_{y^2}^\infty dy$$

$$\rightarrow \frac{1}{2}\int_0^\infty \left[-ye^{-t/y}\right]_{y^2}^\infty dy$$

$$\rightarrow -\frac{1}{2}\int_0^\infty \left[y\left[e^{-\infty} - e^{-y^2/y}\right]\right] dy$$

$$\rightarrow -\frac{1}{2}\int_0^\infty y\left[0 - e^{-y}\right] dy$$

$$\rightarrow \frac{1}{2}\int_0^\infty y e^{-y}\,dy$$

$$\rightarrow \frac{1}{2}\left[y\frac{e^{-y}}{-1}\right]_0^\infty - \int_0^\infty 1.\frac{e^{-y}}{-1}\,dy$$

$$\rightarrow \frac{1}{2}\left[-ye^{-y}\right]_0^\infty + \int_0^\infty e^{-y}\,dy$$

$$\rightarrow \frac{1}{2}\left[\left[-\infty e^{-\infty} - 0\right] + \left[\frac{e^{-y}}{-1}\right]_0^\infty\right]$$

$$\rightarrow \frac{1}{2}\left[0 - 0\right] - \left[e^{-\infty} - e^0\right]$$

$$\rightarrow \frac{1}{2}\left[0 - \left[0 - 1\right]\right]$$

$$= \frac{1}{2} \quad \underline{\underline{\text{Ans}}}$$

Q7 Show that the first four terms of Maclaurin's series Expansion of $e^{ax}\cos by$ are:

$$1 + ax + \frac{1}{\lfloor 2}(a^2x^2 - b^2y^2) + \frac{1}{\lfloor 3}(a^3x^3 - 3ab^2xy^2)$$

Soln The Maclaurin series Expansion of $e^{ax}\cos(by)$ can be written by Combining the Maclaurin series for e^{ax} and $\cos(by)$.

Let $ax = u$ and $by = v$

so Expansion's of e^u and $\cos(v)$ are given as:

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots$$

$$\cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} + \cdots$$

Substituting values of 'u' and 'v' back into above Expansions

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots$$

$$\cos by = 1 - \frac{(by)^2}{2!} + \frac{(by)^4}{4!} - \frac{(by)^6}{6!} + \cdots$$

Multiplying both the series

$$e^{ax}\cos by = \left(1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots\right)\left(1 - \frac{(by)^2}{2!} + \frac{(by)^4}{4!} - \frac{(by)^6}{6!} + \cdots\right)$$

$$= 1 + ax + \frac{1}{2!}(a^2x^2 - b^2y^2) + \frac{1}{3!}(a^3x^3 - 3ab^2xy^2) + \cdots$$

Hence these are Ist four terms of series

Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ over the region $x \geq 0$

$y \geq 0$, $z \geq 0$, $x+y+z \leq 1$

Soln Since $x+y+z \leq 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$

$\therefore$ $x \leq 1$, $x+y \leq 1$, $x+y+z \leq 1$

$\Rightarrow$ $x \leq 1$, $y \leq 1-x$, $z \leq 1-x-y$

$\therefore$ $V = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y\}$

$$\therefore \iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_0^1 \left\{ \int_0^{1-x} \left(\int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3} \right) dy \right\} dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} \left(\int_0^{1-x-y} (x+y+z+1)^{-3} dz \right) dy \right\} dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2} \right]_0^{1-x-y} dy \right\} dx$$

$$= -\frac{1}{2} \int_0^1 \left\{ \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2} \right] dy \right\} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4} y - \frac{(x+y+1)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\left(\frac{1-x}{4} + \frac{1}{2} \right) - \left(0 + \frac{1}{x+1} \right) \right] dx$$

$$= -\frac{1}{8} \int_0^1 \left(1-x+2-\frac{4}{x+1} \right) dx$$

$$= -\frac{1}{8} \int_0^1 \left(-x - \frac{4}{x+1} + 3 \right) dx$$

$$= -\frac{1}{8} \left[-\frac{x^2}{2} - 4\log(1+x) + 3x \right]_0^1$$

$$= -\frac{1}{8}\left[\left(-\frac{1}{2} - 4\log 2 + 3\right) - (0 - 4\log 1 + 0)\right]$$

$$= -\frac{1}{8}\left[\frac{5}{6} - 4\log 2\right]$$

$$= \frac{1}{2}\log 2 - 5/16$$

Hence

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$$