December 2019 CSE/IT (BTAM104-18) Maths-1 Paper Solution

Section-A

Sol 1. $f(x) = 2 + (x-1)^{\frac{2}{3}}, \; x \in [0,2]$

$f'(x) = \frac{2}{3}(x-1)^{\frac{-1}{3}}$ does not exist for

$x = 1$ in $(0,2)$.

Therefore $f(x)$ is not differentiable for all

$x \in (0,2)$

⇒ Rolle's theorem is not applicable to

$f(x) = 2 + (x-1)^{\frac{2}{3}}, \; x \in [0,2]$

Sol 2. Beta function:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Sol 3. $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2 \sin x} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3\left(\frac{\sin x}{x}\right)} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$

(because $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$)

It is $\frac{0}{0}$ form. So applying L'Hospital's rule, we get

$\lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2} = \lim_{x\to 0} \frac{-x\sin x}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{3x}$

It is $\frac{0}{0}$ form. So applying L'Hospital's rule, we get

$\lim_{x\to 0} \frac{-\cos x}{3} = \frac{-1}{3}$

Sol 4. Given $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$

$$\xrightarrow{yields} x + 3 = 0 \xrightarrow{yields} x = -3$$

$$2y + x = -7 \xrightarrow{yields} y = \frac{1}{2}(-7 - x) = \frac{1}{2}(-7 + 3) = -2$$

$$z - 1 = 3 \xrightarrow{yields} z = 4$$

$$4a - 6 = 2a \xrightarrow{yields} 2a = 6 \xrightarrow{yields} a = 3$$

Hence the solution is

$$x = -3, y = -2, z = 4, a = 3$$

Sol 5. $A = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$

Cofactors of elements a_{ij} in A are

$$c_{11} = (-1)^{1+1}(0) = 0$$

$$c_{12} = (-1)^{1+2}(-2) = 2$$

$$c_{21} = (-1)^{2+1}(-1) = 1$$

$$c_{22} = (-1)^{2+2}(1) = 1$$

Cofactor matrix of A is $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

$$\therefore adj. A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

Sol 6. Let V(F) be a vector space. An ordered set B = {v_1, v_2, . . . , v_p} of vectors in V is called a basis for V if

(a)B is a linearly independent set, and

(b) B spans V, that is, V = Linear Span {v_1, v_2, . . . , v_p} i.e. every element of V is expressible as a linear combination of elements of B.

Sol 7. **<u>Rank Nullity Theorem or Sylvester's Law of Nullity</u>**

Let V(F) and W(F) be two vector spaces and $T: V \rightarrow W$ be a linear transformation. If V is of dimension n, then

$$Rank(T) + Nullity(T) = dim(V)$$

i.e. $\rho(T) + \nu(T) = n$

Sol 8. Two properties of Eigen Values are:

(i) Atleast one Eigen Value of every singular matrix is zero.

(ii) A square matrix 'A' and its transpose have the same set of Eigen values.

Sol 9. **Symmetric matrix:** A square matrix is called symmetric matrix if $A = A^T$

i.e.$a_{ij} = a_{ji}$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

Hence $A = A^T$. $\therefore A$ is symmetric matrix.

Sol 10. Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

Trace of given matrix A is = sum of diagonal matrix = 2+3 = 5

Since sum of latent roots of a matrix is equal to trace of that matrix, therefore

sum of latent roots given matrix A = 5

Now $|A| = 6 - 2 = 4$

Since product of latent roots of a matrix is equal to determinant of that matrix, therefore

product of latent roots given matrix A = 4

Section-B

Sol 11 (a) Here $f(x) = sin^{-1}x$

	$f(0) = sin^{-1}0 = 0$
$f'(x) = \frac{1}{\sqrt{1-x^2}}$	$f'(0) = \frac{1}{\sqrt{1-0^2}} = 1$
$f''(x) = \frac{-(-2x)}{2(1-x^2)^{3/2}} = \frac{x}{(1-x^2)^{3/2}}$	$f''(0) = 0$
$f'''(x) = \frac{(1-x^2)^{3/2} - x\frac{3}{2}\sqrt{1-x^2}(-2x)}{(1-x^2)^3}$ $= \frac{1-x^2+3x^2}{(1-x^2)^{5/2}} = \frac{1+2x^2}{(1-x^2)^{5/2}}$	$f'''(0) = \frac{1}{(1-0)^{5/2}} = 1$

By Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$sin^{-1}x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \cdots = x + \frac{x^3}{3!} + \cdots$$

Sol 11 (b) $\lim_{x\to a}\left(\frac{x^a - a^x}{x^x - a^a}\right)$ It is $\frac{0}{0}$ form. Applying L'Hospital's rule

$$\lim_{x\to a}\left(\frac{ax^{a-1} - a^x \log a}{x^x(1+\log x) - 0}\right)$$

$$\left(\begin{array}{c} Because\ let\ y = x^x \xrightarrow{yields} \log y = x\log x \xrightarrow{yields} \frac{1}{y}\frac{dy}{dx} = (1+\log x) \xrightarrow{yields} \frac{dy}{dx} = y(1+\log x) \\ \xrightarrow{yields} \frac{dy}{dx} = x^x(1+\log x) \end{array}\right)$$

$$= \left(\frac{aa^{a-1} - a^a \log a}{a^a(1+\log a)}\right) = \left(\frac{a^a - a^a \log a}{a^a(1+\log a)}\right) = \frac{a^a(1-\log a)}{a^a(1+\log a)} = \frac{(1-\log a)}{(1+\log a)}$$

Sol 12 (a) Given integral is $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^4 = z \xrightarrow{yields} 4x^3dx = dz \xrightarrow{yields} dx = \frac{dz}{4x^3} \xrightarrow{yields} dx = \frac{dz}{4z^{3/4}}$ and

As $x = 0 \xrightarrow{yields} z = 0$ and as $x = 1 \xrightarrow{yields} z = 1$

$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \frac{dz}{4z^{3/4}\sqrt{1-z}} = \frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{-1/2}dz = \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)$

$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$ $\left(because\ \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$

$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}$ $\left(because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$

$$= \frac{\sqrt{\pi}}{4}\left[\frac{\Gamma\left(\frac{1}{4}\right)}{\sqrt{2}\pi/\Gamma\left(\frac{1}{4}\right)}\right]$$

$$\left(\begin{array}{c} \textit{because by } \text{Rodrigue's Duplication formula} \\ \Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m) \\ i.e.\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{\pi}}{2^{\frac{1}{2}-1}}\Gamma\left(\frac{2}{4}\right) \\ i.e.\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\sqrt{2}\pi \\ i.e.\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)} \end{array}\right)$$

$$=\frac{\sqrt{2}}{8\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Sol 12 (b). Given $f(x,y)=2(x^2-y^2)-x^4+y^4$

$\frac{\partial f}{\partial x}=4x-4x^3 \, and \quad \frac{\partial f}{\partial y}=-4y+4y^3$

$\therefore r=\frac{\partial^2 f}{\partial x^2}=4-12x^2, \quad t=\frac{\partial^2 f}{\partial y^2}=-4+12y^2, \quad s=\frac{\partial^2 f}{\partial x \partial y}=0$

$$rt-s^2=(4-12x^2)(-4+12y^2)-0=(4-12x^2)(-4+12y^2)$$

$$\frac{\partial f}{\partial x}=0 \quad and \quad \frac{\partial f}{\partial y}=0 \xrightarrow{yields} 4x-4x^3=0 \quad and \quad 4y-4y^3=0$$

$$\xrightarrow{yields} 4x(1-x^2)=0 \quad and \quad 4y(1-y^2)=0$$

$$\xrightarrow{yields} x=0,\pm 1 \; and \; y=0,\pm 1$$

$\therefore$ $(0,0),\ (1,0),(-1,0),(0,1),(0,-1),\ (-1,-1),\ (-1,1),\ (1,-1)$ *and* $(1,1)$ are the stationary points of $f(x,y)$

Point	r	$rt-s^2$	Maxima or Minima
(0,0)		<0	Neither maxima nor minima

$(1,0)$	= -8 < 0	>0	Maxima
$(-1,0)$	= -8 < 0	>0	Maxima
$(0,1)$	= 4 > 0	>0	Minima
$(0,-1)$	= 4 > 0	>0	Minima
$(-1,-1)$		<0	Neither maxima nor minima
$(-1,1)$		<0	Neither maxima nor minima
$(1,-1)$		<0	Neither maxima nor minima
$(1,1)$		<0	Neither maxima nor minima

So minima exists at $(0,1)$ and $(0,-1)$

and maxima exist at $(1,0)$ and $(-1,0)$

Minimum value is $f(0,1) = 2(0^2 - 1^2) - 0^4 + 1^4 = -1$

and $f(0,-1) = 2(0^2 - (-1)^2) - 0^4 + (-1)^4 = -1$

Maximum Value is $f(1,0) = 2(1^2 - 0^2) - 1^4 + 0^4 = 1$

And $f(-1,0) = 2((-1)^2 - 0^2) - (-1)^4 + 0^4 = 1$

Sol 13 (a). $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+\mathrm{c} \end{vmatrix}$

(Taking out a, b, c $common\ from\ R_1, R_2\ and\ R_3\ respectively$)

$$\xrightarrow{yields} abc\begin{vmatrix} \frac{1}{a}+1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

(Operating $R_1 \to R_1 + R_2 + R_3$)

$$\xrightarrow{yields} abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

(Taking out $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ $common\ from\ R_1$)

$$\xrightarrow{yields} abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$(Operating\ C_2 \to C_2 - C_1\ and\ C_3 \to C_3 - C_1)$

$$\xrightarrow{yields} abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

(Expanding determinant w.r.t. R_1)

$$\xrightarrow{yields} abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1)$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Sol 13 (b) Given equations are $x + y + z = 1,\ x + 2y + 3z = 6,$

$$x + 3y + 4z = 6$$

By Cramer's rule

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1(8-9) - 1(4-3) + 1(3-2) = -1 - 1 + 1 = -1$$

Here $\Delta = -1 \neq 0$

∴System has unique solution.

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 2 & 3 \\ 6 & 3 & 4 \end{vmatrix} = 1(8-9) - 1(24-18) + 1(18-12) = -1 - 6 + 6$$

$$= -1$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 6 & 3 \\ 1 & 6 & 4 \end{vmatrix} = 1(24-18) - 1(4-3) + 1(6-6) = 6 - 1 + 0 = 5$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 6 \\ 1 & 3 & 6 \end{vmatrix} = 1(12-18) - 1(6-6) + 1(3-2) = -6 - 0 + 1 = -5$$

∴System has unique many solution.

$$x = \frac{\Delta_1}{\Delta} = \frac{-1}{-1} = 1, \;\; y = \frac{\Delta_2}{\Delta} = \frac{5}{-1} = -5, \;\; z = \frac{\Delta_3}{\Delta} = \frac{-5}{-1} = 5$$

Sol 14 (a). $\begin{vmatrix} 2 & 1 & 1 \\ 2 & 0 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 2(0+2) - 1(2+4) + 1(4-0) = 4 - 6 + 4 = 2 \neq 0$

∴ Vectors linearly independent.

Sol 14 (b) $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \xrightarrow{R_1 \to \frac{R_1}{5}} \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} \\ 0 & \frac{-11}{5} & \frac{1}{5} \end{bmatrix}$

$$\xrightarrow{R_3 \to 11R_3} \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} \\ 0 & \frac{-121}{5} & \frac{11}{5} \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & \frac{121}{5} & \frac{-11}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

Since Echelon form contains two non-zero rows, so rank of given matrix is 2.

Sol 15. $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{yields} \begin{vmatrix} 2-\alpha & 0 & -1 \\ 5 & 1-\alpha & 0 \\ 0 & 1 & 3-\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} (2-\alpha)\begin{vmatrix} 1-\alpha & 0 \\ 1 & 3-\alpha \end{vmatrix} - 0 - 1\begin{vmatrix} 5 & 1-\alpha \\ 0 & 1 \end{vmatrix} = 0$$

$$\xrightarrow{yields} (2-\alpha)(1-\alpha)(3-\alpha) - 5 = 0$$

$$\xrightarrow{yields} (\alpha^2 - 3\alpha + 2)(3-\alpha) - 5 = 0$$

$$\xrightarrow{yields} 3\alpha^2 - 9\alpha + 6 - \alpha^3 + 3\alpha^2 - 2\alpha - 5 = 0$$

$$\xrightarrow{yields} -\alpha^3 + 6\alpha^2 - 11\alpha + 1 = 0$$

$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 11\alpha - 1 = 0$ is the characteristic equation of matrix A.

Since Cayley Hamilton theorem states that every square matrix satisfies its own characteristic equation, therefore A satisfies the equation

$$A^3 - 6A^2 + 11A - I = 0$$

Alternate Method

$$L.H.S. = A^2 = A.A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix}$$

Hence $A^3 - 6A^2 + 11A - I$

$$= \begin{bmatrix} 3 & -6 & -19 \\ 35 & -4 & -30 \\ 30 & 13 & 22 \end{bmatrix} - 6\begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} + 11\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-24+22-1 & -6+6+0-0 & -19+30-11-0 \\ 35-90+55-0 & -4-6+11-1 & -30+30+0-0 \\ 30-30+0-0 & 13-24+11-0 & 22-54+33-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= R.H.S.$$

Therefore A satisfies the equation

$$A^3 - 6A^2 + 11A - I = 0$$

Sol16. Let $\begin{bmatrix} a \\ b \end{bmatrix} \epsilon R^2$ be an arbitrary vector. Since $Y = \{(1,0),(0,1)\}^T$ is an ordered basis of R^2, so $\begin{bmatrix} a \\ b \end{bmatrix}$ can be expressed as a linear combination of elements $\{(1,0),(0,1)\}$

Let $\begin{bmatrix} a \\ b \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{yields} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \xrightarrow{yields} \alpha = a \;\; and \;\; \beta = b$

Hence $\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-z \end{bmatrix}$

$$\therefore T\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1+0\\1-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix} \text{ .... (2)}$$

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1+1\\1-0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix} \text{ .... (3)}$$

$$T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0+1\\0-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix} \text{ .... (4)}$$

From (2), (3) and (4), required matrix representation of T is $\begin{bmatrix}1 & 2 & 1\\0 & 1 & -1\end{bmatrix}$

Sol 17(a). **Orthogonal matrix:** A square matrix A is said to be orthogonal if

$$AA^T = A^TA = I.$$

Let $A = \begin{bmatrix}4 & 2 & 1\\6 & 3 & 4\\2 & 1 & 0\end{bmatrix}$

Then $A^T = \begin{bmatrix}4 & 6 & 2\\2 & 3 & 1\\1 & 4 & 0\end{bmatrix}$

Now $AA^T = \begin{bmatrix}4 & 2 & 1\\6 & 3 & 4\\2 & 1 & 0\end{bmatrix}\begin{bmatrix}4 & 6 & 2\\2 & 3 & 1\\1 & 4 & 0\end{bmatrix} =$

$$\begin{bmatrix}16+4+1 & 24+6+4 & 8+2\\24+6+4 & 36+9+16 & 12+3\\8+2 & 12+3 & 4+1\end{bmatrix} = \begin{bmatrix}21 & 34 & 10\\34 & 61 & 15\\10 & 15 & 5\end{bmatrix} \neq \text{I}$$

Hence given matrix A is not an orthogonal matrix.

Sol 17 (b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$

$$\therefore A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & -2 \\ 10 & -2 & 18 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix}$$

Now $\left[\frac{1}{2}(A + A^T)\right]^T = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix} = \frac{1}{2}(A + A^T)$

Hence $\frac{1}{2}(A + A^T)$ is a symmetric matrix.

$$\frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 14 \\ 4 & -14 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix}$$

Now $\left[\frac{1}{2}(A - A^T)\right]^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -7 \\ -2 & 7 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix} =$
$-\frac{1}{2}(A - A^T)$

Hence $\frac{1}{2}(A - A^T)$ is a skew-symmetric matrix.

Since $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & -1 \\ 5 & -1 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 7 \\ 2 & -7 & 0 \end{bmatrix}$$

Is representation of given matrix as the sum of a symmetric matrix and a skew-symmetric matrix.

Sol 18. $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0$

$$\xrightarrow{\text{yields}} \begin{vmatrix} -1-\alpha & 2 & -2 \\ 1 & 2-\alpha & 1 \\ -1 & -1 & -\alpha \end{vmatrix} = 0$$

$$\xrightarrow{yields} (-1-\alpha)[-2\alpha + \alpha^2 + 1] - 2(-\alpha + 1) - 2[-1 + 2 - \alpha] = 0$$

$$\xrightarrow{yields} 2\alpha - \alpha^2 - 1 + 2\alpha^2 - \alpha^3 - \alpha + 2\alpha - 2 - 2 + 2\alpha = 0$$

$$\xrightarrow{\text{yields}} -\alpha^3 + \alpha^2 + 5\alpha - 5 = 0$$

$\xrightarrow{\text{yields}} \alpha^3 - \alpha^2 - 5\alpha + 5 = 0 \xrightarrow{\text{yields}} \alpha = 1, \sqrt{5}, -\sqrt{5}$ *are Eigen values of given matrix.*

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *be the Eigen vector of A corresponding to Eigen value* $\alpha = 1$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{\text{yields}} [A - (1)I]X_1 = 0$$

$$\xrightarrow{\text{yields}} \begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\text{yields}} -2x + 2y - 2z = 0,\ x + y + z = 0,\ -x - y - z = 0$$

$$\xrightarrow{\text{yields}} x - y + z = 0, \qquad x + y + z = 0$$

From first two equations,

$$\frac{x}{\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}} = \frac{y}{\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}} = \frac{z}{\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}} \xrightarrow{\text{yields}} \frac{x}{-2} = \frac{y}{0} = \frac{z}{2} \xrightarrow{\text{yields}} \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

$\therefore\ X_1 = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$ *or* $\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$ *is the Eigen vector of A corresponding to Eigen value* $\alpha = 1.$

Let $X_2 = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$ *be the Eigen vector of A corresponding to Eigen value* $\alpha = \sqrt{5}.$

$$\therefore\ [A - \alpha I]X_2 = 0 \xrightarrow{\text{yields}} \left[A - \left(\sqrt{5}\right)I\right]X_2$$

$$= 0 \xrightarrow{\text{yields}} \begin{bmatrix}-1-\sqrt{5} & 2 & -2\\ 1 & 2-\sqrt{5} & 1\\ -1 & -1 & -\sqrt{5}\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\xrightarrow{\text{yields}} \left(-1-\sqrt{5}\right)x + 2y - 2z = 0\,,\ x + \left(2-\sqrt{5}\right)y + z = 0,\ -x - y - \sqrt{5}z = 0$

From last two equations i.e.

$x + \left(2-\sqrt{5}\right)y + z = 0,\ x + y + \sqrt{5}z = 0,$

we get

$$,\ \frac{x}{\begin{vmatrix}\left(2-\sqrt{5}\right) & 1\\ 1 & \sqrt{5}\end{vmatrix}} = \frac{y}{\begin{vmatrix}1 & 1\\ \sqrt{5} & 1\end{vmatrix}} = \frac{z}{\begin{vmatrix}1 & \left(2-\sqrt{5}\right)\\ 1 & 1\end{vmatrix}}$$

$$\xrightarrow{\text{yields}} \frac{x}{2\sqrt{5}-5-1} = \frac{y}{1-\sqrt{5}} = \frac{z}{1-2+\sqrt{5}}$$

$$\xrightarrow{\text{yields}} \frac{x}{2\sqrt{5}-6} = \frac{y}{1-\sqrt{5}} = \frac{z}{-1+\sqrt{5}}$$

$$\xrightarrow{\text{yields}} \frac{x}{-2\sqrt{5}+6} = \frac{y}{-1+\sqrt{5}} = \frac{z}{1-\sqrt{5}}$$

$$\xrightarrow{\text{yields}} \frac{x}{\left(\sqrt{5}-1\right)^2} = \frac{y}{\sqrt{5}-1} = \frac{z}{1-\sqrt{5}}$$

$$\xrightarrow{\text{yields}} \frac{x}{\sqrt{5}-1} = \frac{y}{1} = \frac{z}{-1}$$

$\therefore\ X_2 = \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$ *is the Eigen vector of A corresponding to Eigen value* $\alpha = \sqrt{5}$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *be the Eigen vector of A corresponding to Eigen value* $\alpha = -\sqrt{5}$.

$$\therefore\ [A-\alpha I]X_3 = 0 \xrightarrow{\text{yields}} \left[A-\left(-\sqrt{5}\right)I\right]X_3$$

$$= 0 \xrightarrow{\text{yields}} \begin{bmatrix} -1+\sqrt{5} & 2 & -2 \\ 1 & 2+\sqrt{5} & 1 \\ -1 & -1 & \sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{\text{yields}} \left(-1+\sqrt{5}\right)x + 2y - 2z = 0,\ x + \left(2+\sqrt{5}\right)y + z = 0,\ -x - y + \sqrt{5}z = 0$

From first two equations,

$$\frac{x}{\begin{matrix} 2 & -2 \\ \left(2+\sqrt{5}\right) & 1 \end{matrix}} = \frac{y}{\begin{matrix} -2 & \left(-1+\sqrt{5}\right) \\ 1 & 1 \end{matrix}} = \frac{z}{\begin{matrix} \left(-1+\sqrt{5}\right) & 2 \\ 1 & \left(2+\sqrt{5}\right) \end{matrix}}$$

$$\xrightarrow{\text{yields}} \frac{x}{6+2\sqrt{5}} = \frac{y}{-1-\sqrt{5}} = \frac{z}{1+\sqrt{5}}$$

$$\xrightarrow{\text{yields}} \frac{x}{\left(1+\sqrt{5}\right)^2} = \frac{y}{-\left(1+\sqrt{5}\right)} = \frac{z}{1+\sqrt{5}}$$

$$\xrightarrow{\text{yields}} \frac{x}{(1+\sqrt{5})} = \frac{y}{-1} = \frac{z}{1}$$

$\therefore\ X_3 = \begin{bmatrix} 1+\sqrt{5} \\ -1 \\ 1 \end{bmatrix}$ *is the Eigen vector of A corresponding to Eigen value*
$\alpha = -\sqrt{5}.$

$\therefore$ Modal Matrix $P = \begin{bmatrix} 1 & \sqrt{5}-1 & 1+\sqrt{5} \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

$$|P| = \begin{vmatrix} 1 & \sqrt{5}-1 & 1+\sqrt{5} \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 1(1-1) - 0(\sqrt{5}-1+1+\sqrt{5}) - 1(-\sqrt{5}+1-1-\sqrt{5})$$

$= 0 - 0 + 2\sqrt{5} = 2\sqrt{5} \neq 0.$

Hence vectors are linearly independent and the given matrix is

Diagonalizable.

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = 1$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} \sqrt{5}-1 & 1+\sqrt{5} \\ -1 & 1 \end{vmatrix} = -2\sqrt{5}$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1+\sqrt{5} \\ -1 & 1 \end{vmatrix} = 2+\sqrt{5}$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & \sqrt{5}-1 \\ -1 & -1 \end{vmatrix} = 2-\sqrt{5}$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} \sqrt{5}-1 & 1+\sqrt{5} \\ 1 & -1 \end{vmatrix} = -2\sqrt{5}$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1+\sqrt{5} \\ 0 & -1 \end{vmatrix} = 1$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & \sqrt{5}-1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore adj.P = \begin{bmatrix} 0 & 1 & 1 \\ -2\sqrt{5} & 2+\sqrt{5} & 2-\sqrt{5} \\ -2\sqrt{5} & 1 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} adj.P = \frac{1}{2\sqrt{5}}\begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}$$

Diagonal Matrix = D = $P^{-1}AP$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{5}-1 & 1+\sqrt{5} \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} 1 & (1+2+2-\sqrt{5}) & (-1-2-2-\sqrt{5}) \\ 0 & (\sqrt{5}-1+2-1) & (\sqrt{5}+1-2+1) \\ -1 & (-\sqrt{5}+1-1) & (-\sqrt{5}-1+1) \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix} 0 & -2\sqrt{5} & -2\sqrt{5} \\ 1 & 2+\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} 1 & 5-\sqrt{5} & -5-\sqrt{5} \\ 0 & \sqrt{5} & \sqrt{5} \\ 1 & -\sqrt{5} & -\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}}\begin{bmatrix} 2\sqrt{5} & -10+10 & -10+10 \\ 0 & (5-\sqrt{5}+2\sqrt{5}+5-\sqrt{5}) & (-5-\sqrt{5}+2\sqrt{5}+5-\sqrt{5}) \\ 0 & (5-\sqrt{5}+2\sqrt{5}-5-\sqrt{5}) & (-5-\sqrt{5}+2\sqrt{5}-5-\sqrt{5}) \end{bmatrix}$$

$$
= \frac{1}{2\sqrt{5}} \begin{bmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}
$$