B Tech (ECE/ME)
Subject Code - BTAM 101-18

Section - A

1. Test the convergence of the following series:

$$\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots$$

Solution: Consider $u_n = \frac{(n+1)!}{3^n}$

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{3^n} \times \frac{3^{n+1}}{(n+2)!}$$

$$= 3\frac{(n+1)!}{(n+2)(n+1)!} = \frac{3}{n+2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3}{n+2} = 0 < 1$$

$\therefore$ By Ratio Test, Σu_n is divergent

2. State Raabe's Test

Solution: If Σu_n is positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = r \text{ then}$$

a) series converges if $r > 1$

b) series diverges if $r < 1$

3. State Rolle's Theorem

Solution: If a function is

a) continuous in the closed interval $[a, b]$

b) derivable in the open interval (a, b)

c) $f(a) = f(b)$

then there exist at least one real number $c \in (a, b)$ such that $f'(c) = 0$

4. State Lagrange's Mean Value Theorem

Solution: If a function f is

a) continuous in $[a, b]$

b) differentiable in (a, b)

then there exist at least one real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

5. Prove that $\int_0^{\pi/2} \log \tan x \, dx = 0$

Solution: Let $I = \int_0^{\pi/2} \log \tan x \, dx$

$$I = \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right) dx \quad \left[\text{Using } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \log \cot x \, dx$$

$$I = \int_0^{\pi/2} \log\left(\frac{1}{\tan x}\right) dx$$

$$I = -\int_0^{\pi/2} \log \tan x \, dx$$

$$I = -I$$

$$I + I = 0$$

$$2I = 0 \Rightarrow I = 0$$

$$\therefore \int_0^{\pi/2} \log \tan x \, dx = 0$$

6. Evaluate $\int_0^1 \int_0^x e^{y/x} \, dy \, dx$

Solution: $\int_0^1 \left[\int_0^x e^{y/x} dy \right] dx$

$$= \int_0^1 \frac{\left[e^{y/x}\right]_0^x}{1/x} \, dx$$

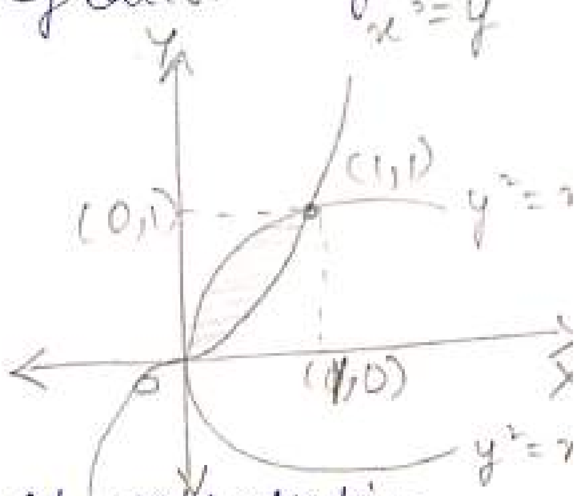
$$= \int_0^1 x \left[e^{x/x} - e^{0/x} \right] dx$$

$$= \int_0^1 x \, (e - 1) \, dx$$

$$= (e-1) \frac{\left[x^2\right]_0^1}{2}$$

$$= \frac{e-1}{2}$$

7. Change the order of integration of
$$\int_0^1 \int_{y^2}^{y^{1/3}} f(x,y)\,dx\,dy$$



Solution: $y = 0$, $y = 1$

$y^2 = x$; $y^{1/3} = x \Rightarrow x^3 = y$

Points of intersection are $(0,0)$ and $(1,1)$

After changing the order of integration

$$\int_0^1 \int_{y^2}^{y^{1/3}} f(x,y)\,dx\,dy = \int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\,dx\,dy$$

8. Find the first order derivative of

$$z = x^3 + y^3 - 3axy$$

Solution: $z = x^3 + y^3 - 3axy$

Differentiating both sides partially w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(3axy)$$

$$\frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay$$

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay$$

Differentiating z partially w.r.t. y

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(3axy)$$

$$\frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax$$

$$\frac{\partial z}{\partial y} = 3y^2 - 3ax$$

9. Find the rank of the following matrix

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution: $R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix}$$

$R_3 \to -\frac{1}{5} R_3$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -1 & -4 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

which is in echelon form.

number of non-zero rows $= 3$, so

rank of matrix is 3

10. Find the determinant of the following matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 1 & 4 & 7 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 1 & 4 & 7 \end{vmatrix}$$

Expanding it along first row

$$= 1\begin{vmatrix} 3 & 6 \\ 4 & 7 \end{vmatrix} - 2\begin{vmatrix} 2 & 6 \\ 1 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= 1(21-24) - 2(14-6) + 5(8-3)$$

$$= -3 - 16 + 25$$

$$= 6$$

Section - B

11. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, Show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

Solution: $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

$$\frac{\partial u}{\partial y} = x^2 \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right) + (-y^2)\frac{\partial}{\partial y}\left(\tan^{-1}\frac{x}{y}\right) + \tan^{-1}\frac{x}{y} \cdot \frac{\partial}{\partial y}(-y^2)$$

$$= x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2} + \tan^{-1}\frac{x}{y} \cdot -2y$$

$$\frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x^3+xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2}$$

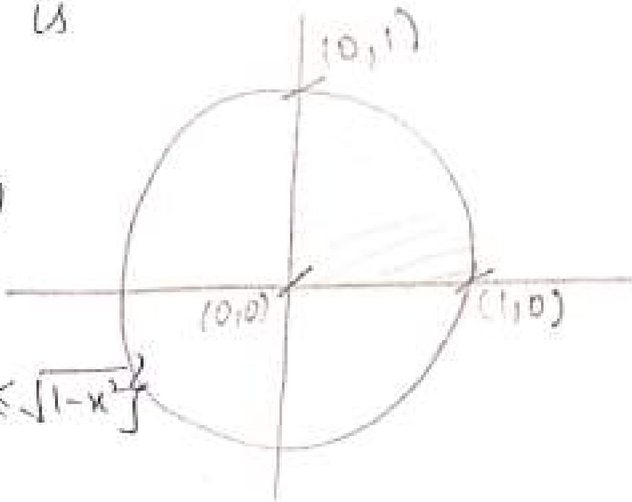
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

Hence proved

12. Evaluate $\iint \frac{xy}{(1-y^2)^{1/2}}\,dx\,dy$ over the first quadrant of the circle $x^2+y^2=1$

Solution: equation of circle is

$$x^2+y^2=1$$



The region of integration is given by

$\{(x,y): 0 \leq x \leq 1 \text{ and } 0 \leq y \leq \sqrt{1-x^2}\}$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{x\,y}{\sqrt{1-y^2}}\, dy\, dx$$

$$= \int_0^1 x \int_0^{\sqrt{1-x^2}} y\,(1-y^2)^{-1/2}\, dy\, dx$$

$$= \frac{-1}{2}\int_0^1 x \int_0^{\sqrt{1-x^2}} -2y\,(1-y^2)^{-1/2}\, dy\, dx$$

$$= -\frac{1}{2}\int_0^1 x \left[\frac{(1-y^2)^{-1/2+1}}{-1/2+1}\right]_0^{\sqrt{1-x^2}} dx$$

$$= -\int_0^1 x\left[(1-1+x^2)^{1/2} - 1\right] dx$$

$$= -\int_0^1 x\,(x-1)\, dx$$

$$= -\int_0^1 x^2 - x \;dx$$

$$= -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

$$= -\frac{1}{3} + \frac{1}{2}$$

$$= \frac{-2+3}{6}$$

$$= \frac{1}{6}$$

13. Test the convergence of the following series:

$$\sum \frac{4.7.\cdots(3n+1)\,x^n}{n!}$$

Solution: Let $a_n = \frac{4.7.10.\cdots(3n+1)}{n!} . x^n$

$$a_{n+1} = \frac{4.7.10.\cdots(3n+4)}{n!} . x^n$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)}{3n+4} \cdot \frac{1}{x} = \frac{1+\frac{1}{n}}{3+\frac{4}{n}} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3+\frac{4}{n}} \cdot \frac{1}{x} = \frac{1}{3x}$$

By Ratio Test, series converges if $\frac{1}{3x} > 1$ i.e. $x < \frac{1}{3}$ and diverges if $x > \frac{1}{3}$

For $x = \frac{1}{3}$ Ratio Test fails

$$\frac{a_n}{a_{n+1}} = \frac{\left(1+\frac{1}{n}\right)}{3+\frac{4}{n}} . 3 = \frac{1+\frac{1}{n}}{3\left(1+\frac{4}{3n}\right)} . 3 = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+O\left(\frac{1}{n^2}\right)\right)$$

$$= 1 - \frac{1}{3n} + O\left(\frac{1}{n^2}\right)$$

Comparing it with $\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$

By Gauss Test, $\mu = -\frac{1}{3} < 1$, series is divergent

$\therefore$ series converges for $x < \frac{1}{3}$ and diverges for $x \geq \frac{1}{3}$

14. Verify if the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal and hence find its inverse

Solution: $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$$A^t = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$AA^t = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AA^t = I$$

$\therefore$ A is orthogonal matrix

$$AA^t = I$$

Pre multiplying by A^{-1} on both sides

$$A^{-1}AA^t = A^{-1}\cdot I$$

$$A^t = A^{-1}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

Section - C

15. Find the maximum and minimum value of $x^3 + y^3 - 3axy$

Solution: Let $f(x,y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$\frac{\partial f}{\partial x} = 0$

$x^2 - ay = 0$

$ay = x^2$

$y = \frac{x^2}{a}$

$\frac{\partial f}{\partial y} = 0$

$y^2 - ax = 0$

$\frac{x^4}{a^2} - ax = 0$

$x^4 - a^3 x = 0$

$x(x^3 - a^3) = 0$

$x = 0, a$

$y = 0, a$

$\therefore$ critical points are $(0,0)$ and (a,a)

$A = \frac{\partial^2 f}{\partial x^2} = 6x$, $B = \frac{\partial^2 f}{\partial x \partial y} = -3a$, $C = \frac{\partial^2 f}{\partial y^2} = 6y$

$$AC - B^2 = (6x)(6y) - (-3a)^2$$
$$= 36xy - 9a^2$$

At $(0,0)$

$AC - B^2 = 0 - 9a^2 = -9a^2 < 0$

$\therefore f(x,y)$ has neither maximum nor minimum at $(0,0)$

At (a,a)

$AC - B^2 = 36a^2 - 9a^2 = 27a^2 > 0$

Also $A = 6a$

a) If $a > 0$ then $A = 6a > 0$

$\therefore f(x,y)$ has a minimum value at (a,a)

and minimum value $= f(a,a) = a^3 + a^3 - 3a^3 = -a^3$

b) If $a < 0$ then $A = 6a < 0$

$\therefore f(x,y)$ has a maximum value at (a,a)

and maximum value $= f(a,a) = -a^3$

6) a) Solve the simultaneous equations:

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

olution: In matrix form equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

i.e. $AX = B$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$R_3 \to \frac{1}{2} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\rho(A) = \rho([A|B]) = 3 = n$

$\therefore$ system has unique solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$x + y + z = 3$

$y + 2z = 1$

$z = 0$

$y = 1$

$x + 1 + 0 = 3$

$x = 2$

$\therefore x = 2, \quad y = 1, \quad z = 0$

16) b) Find the inverse of the matrix

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

$$[A/I] = \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to \frac{1}{2} R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & 2 & 1/2 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & 2 & 1/2 & 0 & 0 \\ 0 & -3 & -7 & -2 & 1 & 0 \\ 0 & 1/2 & 2 & -1/2 & 0 & 1 \end{array}\right]$$

$R_2 \to -\frac{1}{3} R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 7/3 & 2/3 & -1/3 & 0 \\ 0 & 1/2 & 2 & -1/2 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - \frac{3}{2} R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 7/3 & 2/3 & -1/3 & 0 \\ 0 & 1/2 & 2 & -1/2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - \frac{1}{2} R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 7/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 5/6 & -5/6 & 1/6 & 1 \end{array}\right]$$

$R_3 \to \frac{6}{5} R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 7/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1 & 1/5 & 6/5 \end{array}\right]$$

$$R_1 \to R_1 + \frac{3}{2} R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4/5 & 9/5 \\ 0 & 1 & 7/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1 & 1/5 & 6/5 \end{array}\right]$$

$$R_2 \to R_2 - \frac{7}{3} R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 4/5 & 9/5 \\ 0 & 1 & 0 & 3 & -4/5 & -14/5 \\ 0 & 0 & 1 & -1 & 1/5 & 6/5 \end{array}\right]$$

$$.A^{-1} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$$

17) a) Find the area of the surface of revolution generated by revolving the curve $x = y^3$ from $y = 0$ to 2

Solution: Required Surface area $= \int_0^2 2\pi (y^3) \frac{ds}{dy}\, dy$

$$= 2\pi \int_0^2 y^3 \frac{ds}{dy}\, dy$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (3y^2)^2} = \sqrt{1 + 9y^4}$$

$$= 2\pi \int_0^2 y^3 \sqrt{1 + 9y^4}\, dy$$

$$= \frac{2\pi}{36} \int_0^2 36y^3 (1+9y^4)^{1/2} dy$$

$$= \frac{\pi}{18} \left[\frac{(1+9y^4)^{3/2}}{3/2} \right]_0^2$$

$$= \frac{2\pi}{18 \times 3} \left[(1+9(2^4))^{3/2} - (1+9(0)^4)^{3/2} \right]$$

$$= \frac{\pi}{27} \left[(145)^{3/2} - 1 \right]$$

17) b) Evaluate $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

Solution: $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

$$= \int_{-1}^{1} \int_{0}^{z} \left[\frac{(x+y+z)^2}{2} \right]_{x+z}^{x+z} dx\, dz$$

$$= \frac{1}{2} \int_{-1}^{1} \int_{0}^{z} \left((2x+2z)^2 - 4x^2 \right) dx\, dz$$

$$= \frac{1}{2} \int_{-1}^{1} \left[2 \cdot \frac{(x+z)^3}{3} - \frac{4}{3} x^3 \right]_0^z dz$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\frac{2}{3} \times 8z^3 - \frac{4}{3} z^3 - \frac{2}{3} z^3 \right) dz$$

$$= \frac{1}{2} \int_{-1}^{1} \left(\frac{16}{3} - \frac{4}{3} - \frac{2}{3} \right) z^3\, dz$$

$$= \frac{11}{3} \int_{-1}^{1} z^3 = 0 \quad \left[\int_{-a}^{a} f(x)\, dx = 0 \text{ if } f(x) \text{ is odd function} \right.$$

18)a) Test the convergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$

Solution: Consider $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$

$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2}$$

$$= \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)}\sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By Ratio Test,

Σu_n converges if $\frac{1}{x^2} > 1$ i.e. $x^2 < 1$

and diverges if $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$

When $x^2 = 1$, Ratio Test fails

$$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

Take $v_n = \dfrac{1}{n^{3/2}}$ By Comparison test and p-test, Σu_n is convergent for $x^2 = 1$

$\therefore$ Σu_n is convergent for $x^2 \leq 1$ and divergent for $x^2 > 1$

18) b) find the Maclaurin's series of $f(x) = \cos x$

Solution: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{\lfloor \underline{n}} x^n$

or

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor \underline{2}} f''(0) + \frac{x^3}{\lfloor \underline{3}} f'''(0) + \cdots$$

$f(x) = \cos x$ $\quad$ $f(0) = 1$

$f'(x) = -\sin x$ $\quad$ $f'(0) = 0$

$f''(x) = -\cos x$ $\quad$ $f''(0) = -1$

$f'''(x) = \sin x$ $\quad$ $f'''(0) = 0$

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$$f(x) = 1 - \frac{x^2}{\lfloor \underline{2}} + \frac{x^4}{\lfloor \underline{4}} - - - - - - -$$

or $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(\lfloor \underline{2n})}$