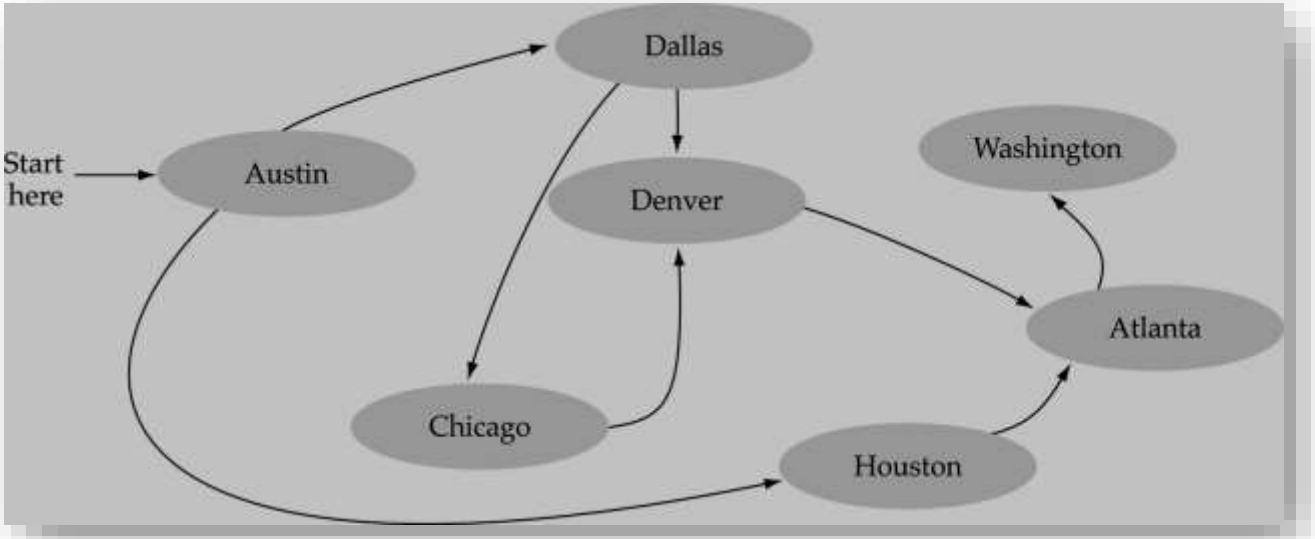
MODULE 5
GRAPHS

What is a graph?

- A data structure that consists of a set of nodes (*vertices*) and a set of edges that relate the nodes to each other
- The set of edges describes relationships among the vertices

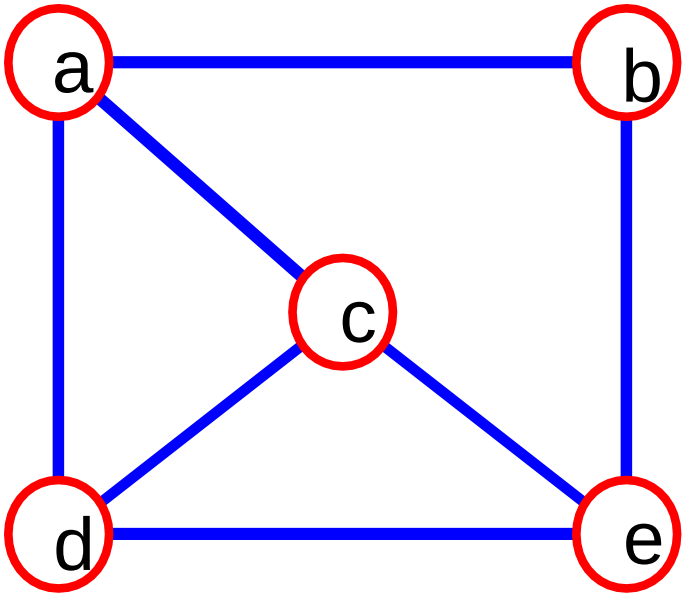


What is a Graph?

- A graph G = (V,E) is composed of:

  V: set of vertices

  E: set of edges connecting the vertices in V
- An edge e = (u,v) is a pair of vertices
- Example:



V= {a,b,c,d,e}

E= {(a,b),(a,c),
(a,d),
(b,e),(c,d),(c,e),
(d,e)}

Formal definition of graphs

- A graph *G* is defined as follows:

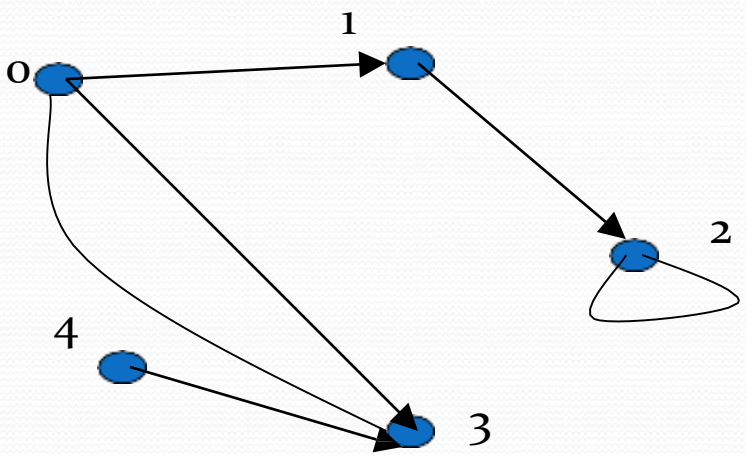
  $G=(V,E)$

  V(G): a finite, nonempty set of vertices

  E(G): a set of edges (pairs of vertices)

Examples of Graphs

- V={0,1,2,3,4}
- E={(0,1), (1,2), (0,3), (3,0), (2,2), (4,3)}



When (x,y) is an edge,
we say that x is *adjacent to* y, and y is *adjacent from* x.

1 is adjacent to 2.
2is not adjacent to 0.
3 is adjacent from 0.

Directed vs. Undirected Graphs

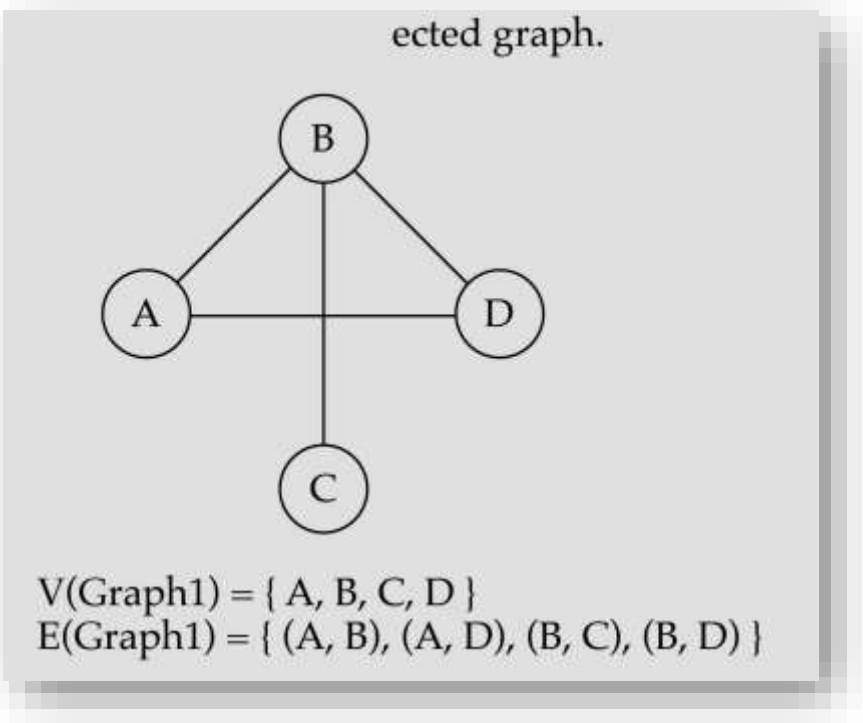
- Undirected edge has no orientation (no arrow head)
- Directed edge has an orientation (has an arrow head)
- Undirected graph – all edges are undirected
- Directed graph – all edges are directed

undirected edge

directed edge

Directed vs. undirected graphs

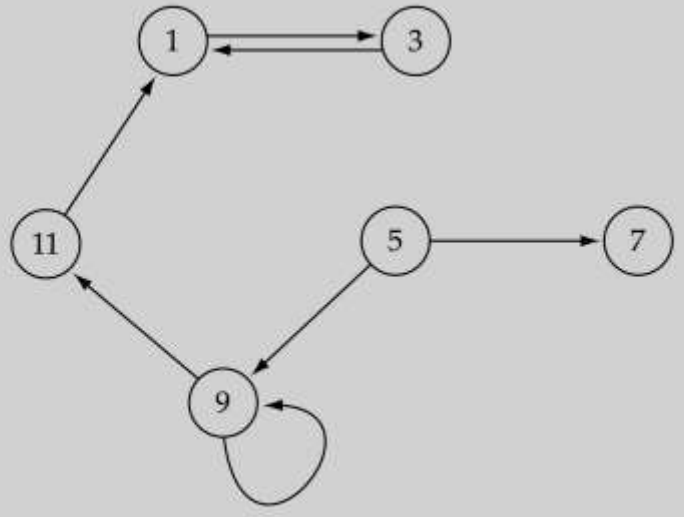
- When the edges in a graph have no direction, the graph is called *undirected*



Directed vs. undirected graphs (cont.)

- When the edges in a graph have a direction, the graph is called *directed* (or *digraph*)

(b) Graph2 is a directed graph.



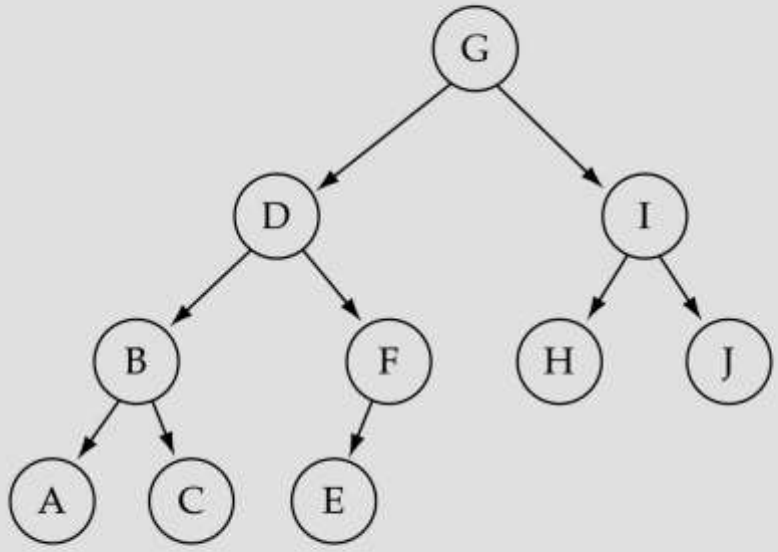
V(Graph2) = { 1, 3, 5, 7, 9, 11 }
E(Graph2) = {(1,3) (3,1) (5,9) (9,11) (5,7) 1), (9, 9), (11, 1) }

Warning: if the graph is directed, the order of the vertices in each edge is important !!

Trees vs graphs

- Trees are special cases of graphs!!

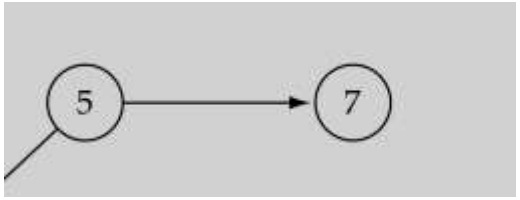
(c) Graph3 is a directed graph.



V(Graph3) = { A, B, C, D, E, F, G, H, I, J }
E(Graph3) = { (G, D), (G, J), (D, B), (D, F) (I, H), (I, J), (B, A), (B, C), (F, E) }

Graph terminology

- Adjacent nodes: two nodes are adjacent if they are connected by an edge

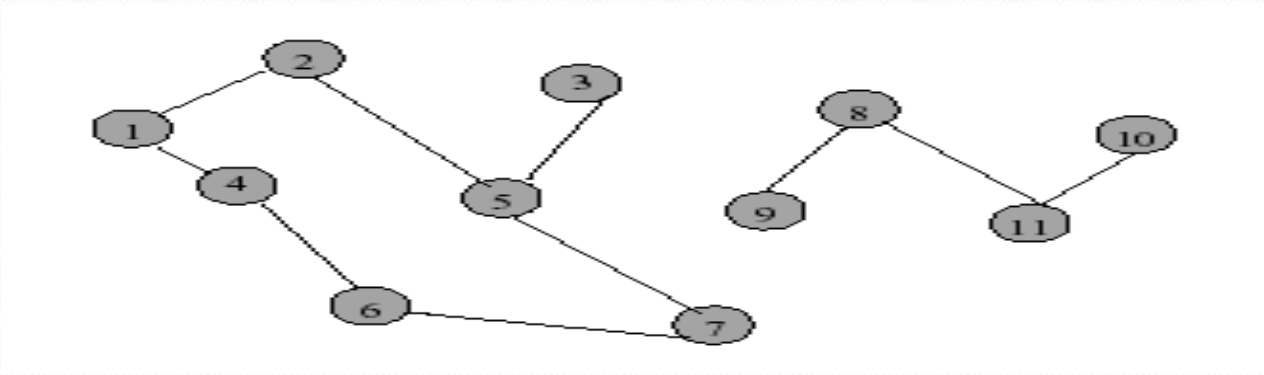


5 is adjacent to 7
7 is adjacent from 5

- Path: a sequence of vertices that connect two nodes in a graph
- Complete graph: a graph in which every vertex is directly connected to every other vertex

Continued…

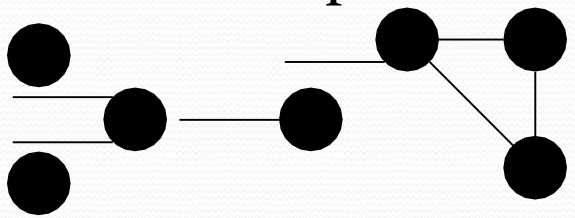
- A cycle is a simple path with the same start and end vertex.
- The **degree** of vertex *i* is the no. of edges incident on vertex *i*.



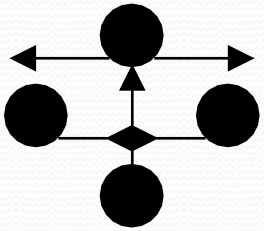
e.g., degree(2) = 2, degree(5) = 3, degree(3) = 1

Continued…

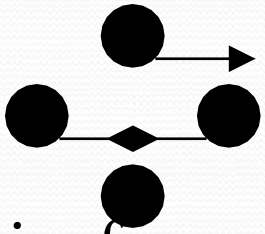
Undirected graphs are *connected* if there is a path between any two vertices



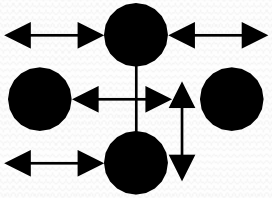
Directed graphs are *strongly connected* if there is a path from any one vertex to any other



Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction*



A *complete* graph has an edge between every pair of vertices

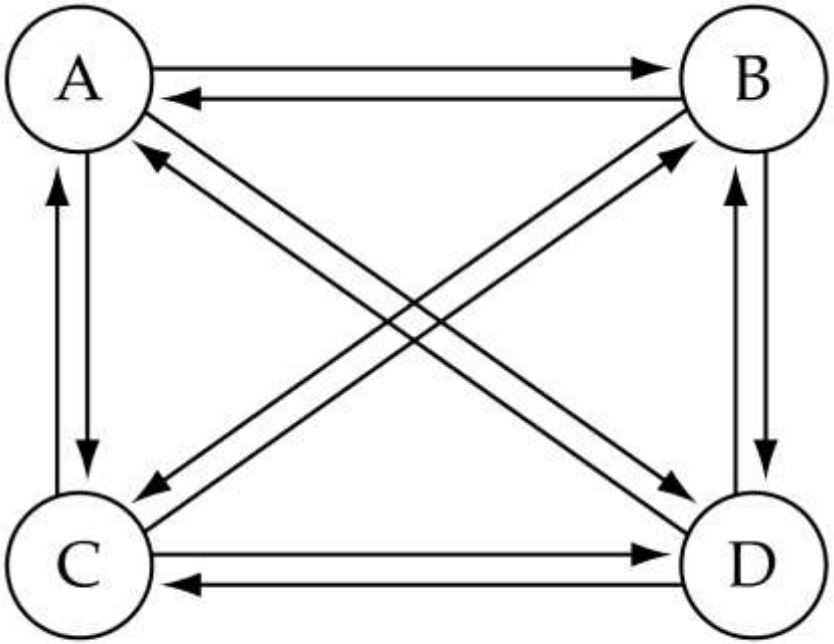


Graph terminology

- What is the number of edges in a complete directed graph with N vertices?

*N * (N-1)*

$O(N^2)$



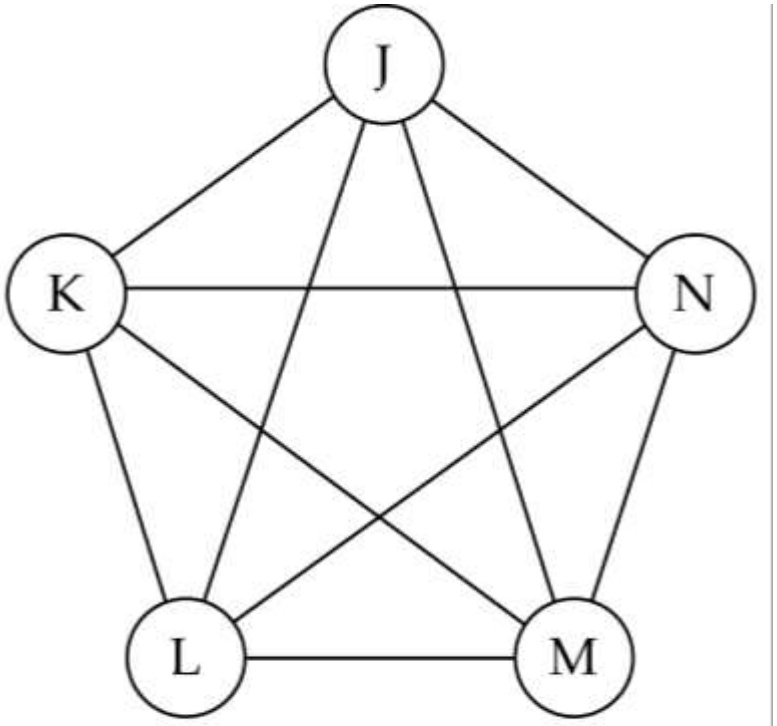
(a) Complete directed graph.

Graph terminology

- What is the number of edges in a complete undirected graph with N vertices?

*N * (N-1) / 2*

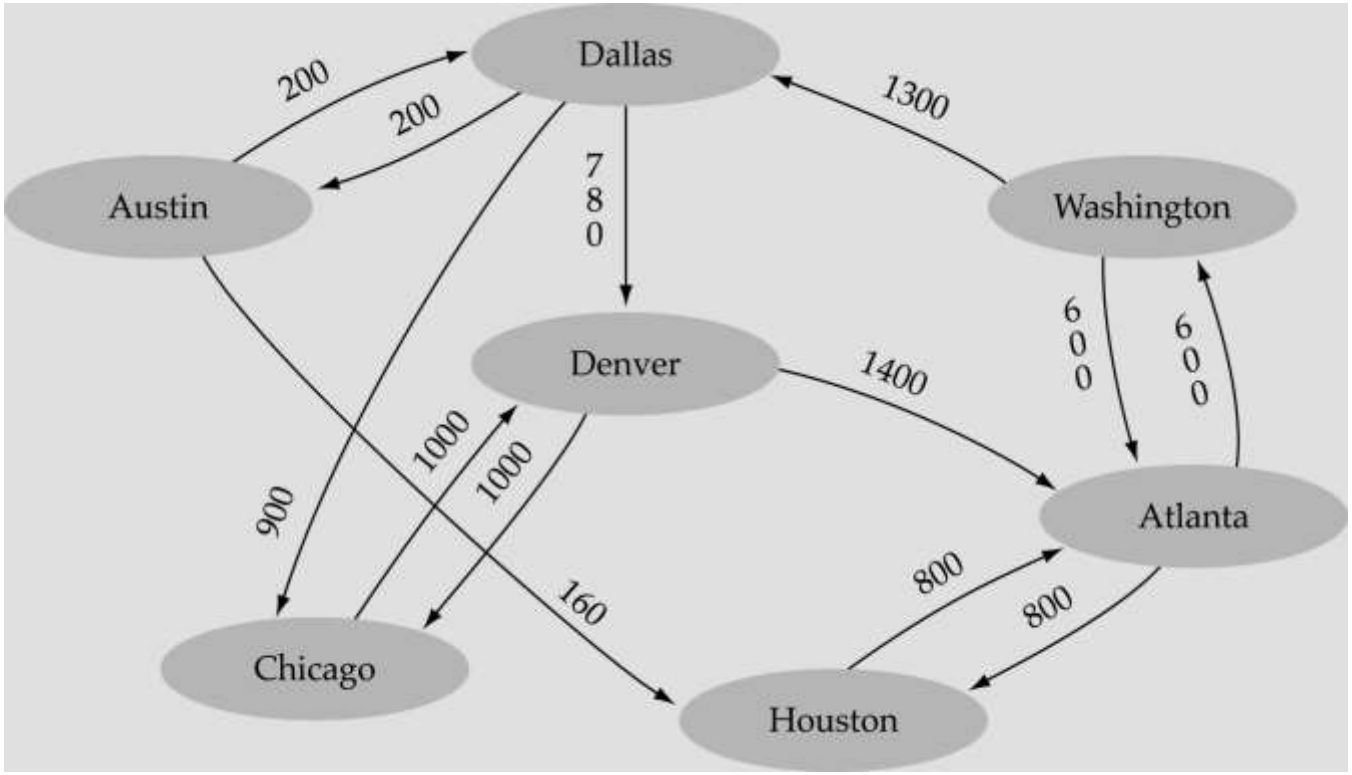
$O(N^2)$

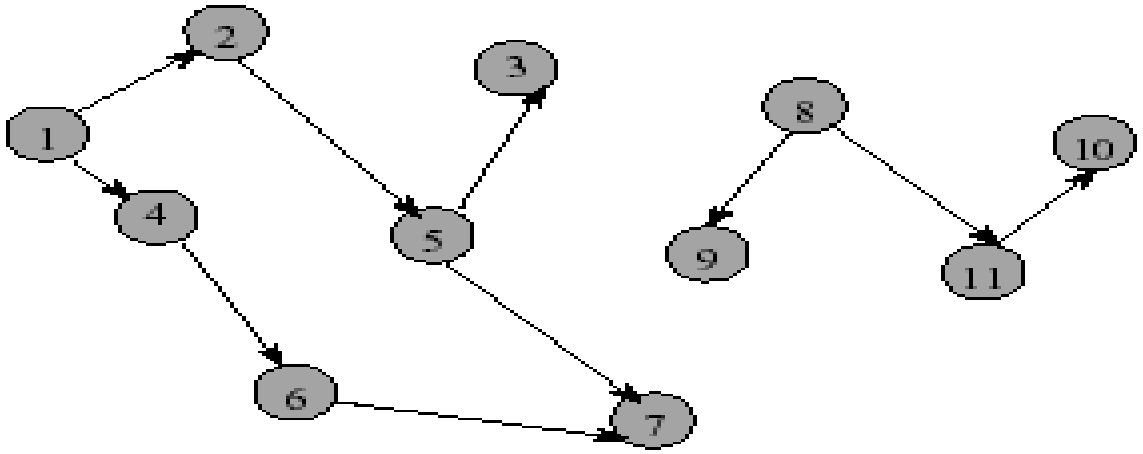


(b) Complete undirected graph.

Graph terminology

- Weighted graph: a graph in which each edge carries a value





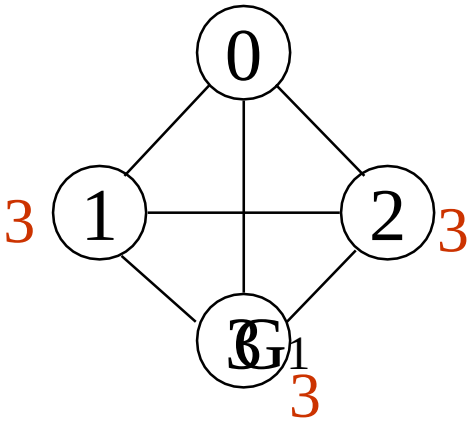
- In-degree of vertex *i* is the number of edges incident to *i* (i.e., the number of incoming edges).

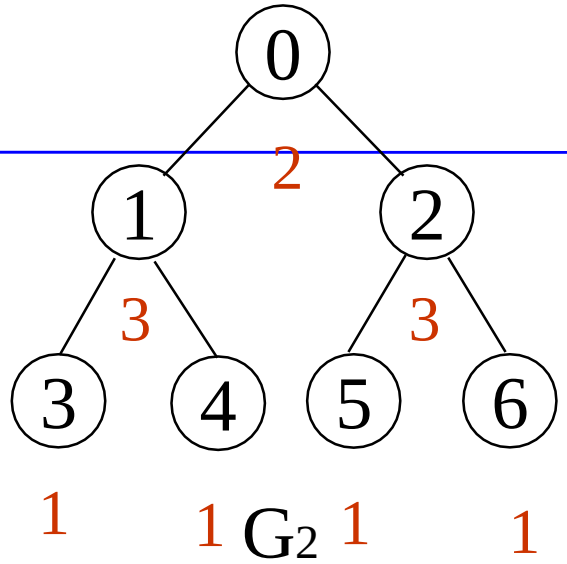
  e.g., indegree(2) = 1, indegree(8) = 0

- Out-degree of vertex *i* is the number of edges incident from *i* (i.e., the number of outgoing edges).

  e.g., outdegree(2) = 1, outdegree(8) = 2

Examples

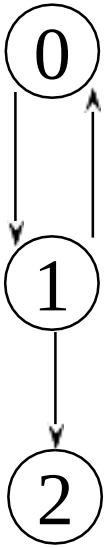




directed graph

in-degree

out-degree



in:1, out: 1

in: 1, out: 2

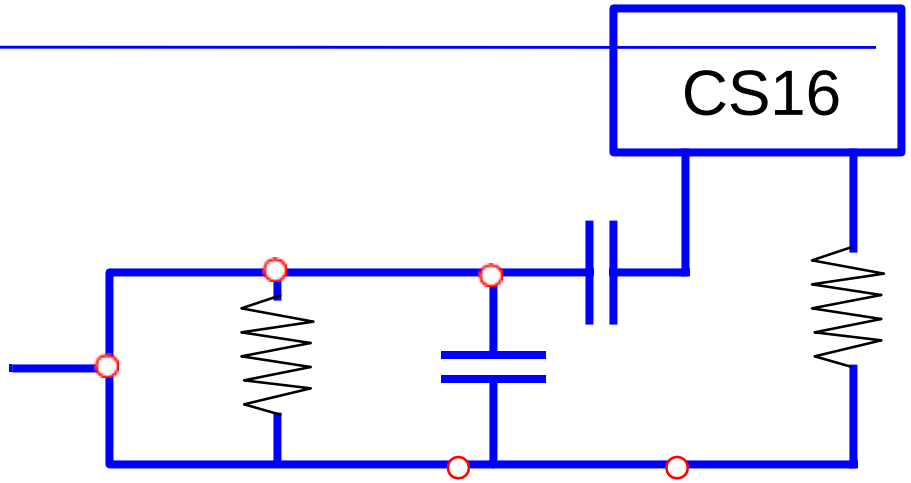
in: 1, out: 0

Continued…

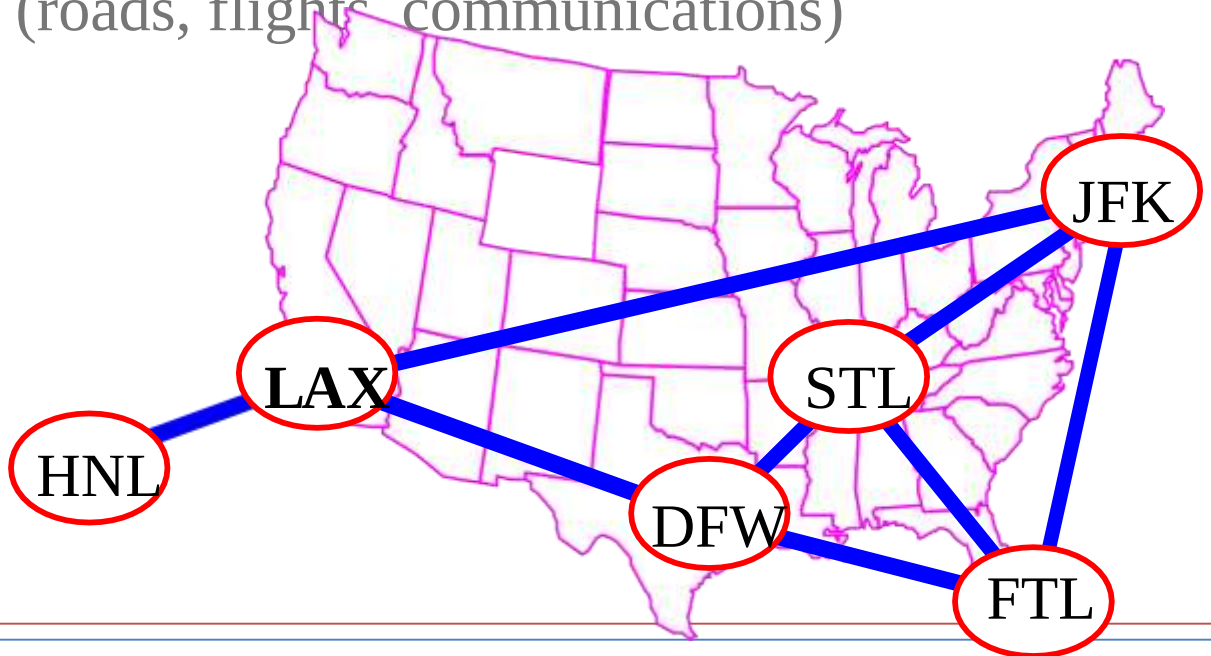
- *Loops*: edges that connect a vertex to itself
- *Paths*: sequences of vertices po, p1, … pm such that each adjacent pair of vertices are connected by an edge
- A simple path is a path in which all vertices, except possibly in the first and last, are different.
- *Multiple Edges*: two nodes may be connected by >1 edge
- *Simple Graphs*: have no loops and no multiple edges

Applications

- electronic circuits



- networks (roads, flights, communications)

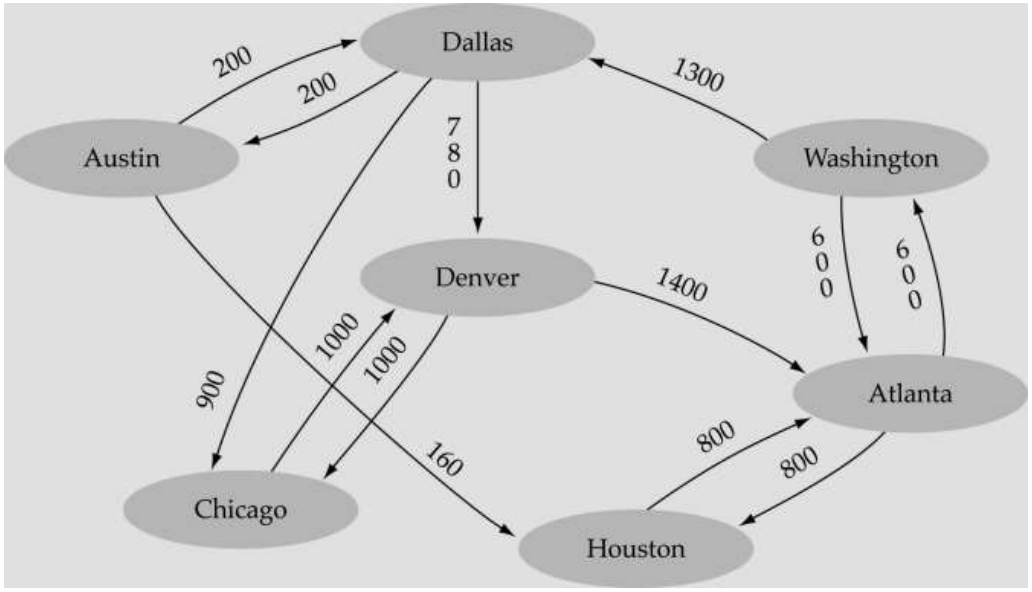


Graph Representation

- For graphs to be computationally useful, they have to be conveniently represented in programs
- There are two computer representations of graphs:
  - Adjacency matrix representation
  - Adjacency lists representation

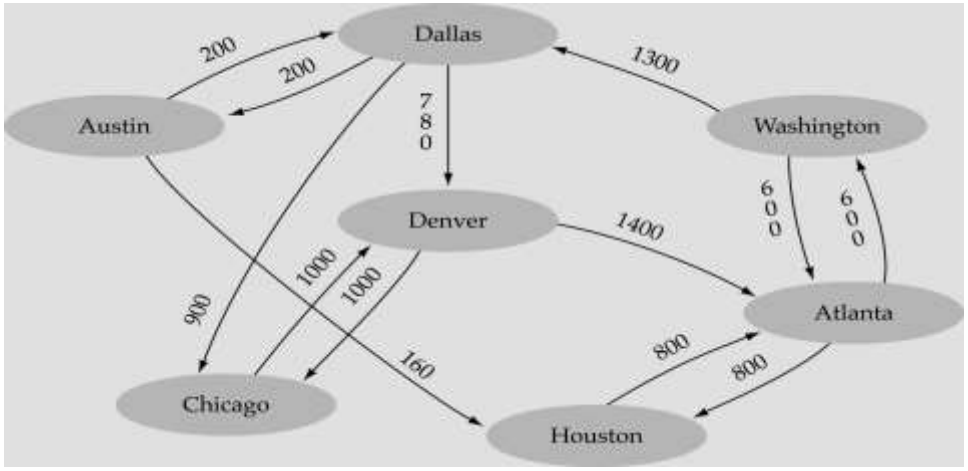
Graph implementation

- Array-based implementation
  - A 1D array is used to represent the vertices
  - A 2D array (adjacency matrix) is used to represent the edges



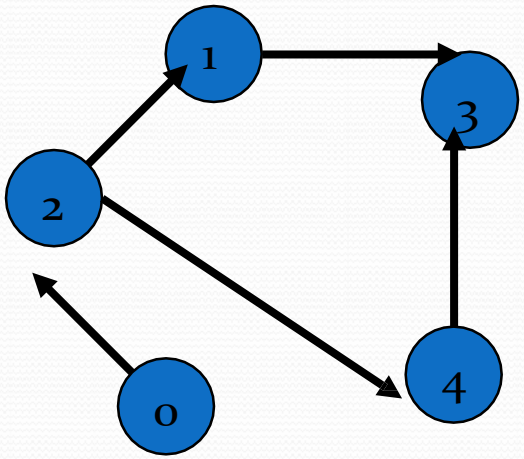
Graph implementation

- Linked-list implementation
  - A 1D array is used to represent the vertices
  - A list is used for each vertex *v* which contains the vertices which are adjacent from *v* (adjacency list)



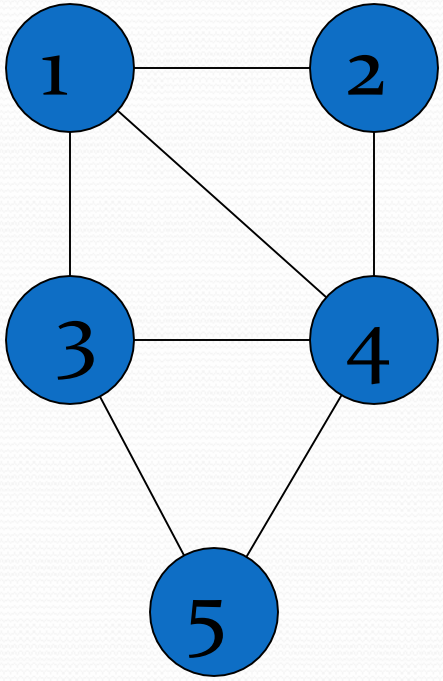
- *Adjacency Matrix*
  - A square grid of boolean values
  - If the graph contains N vertices, then the grid contains N rows and N columns
  - For two vertices numbered I and J, the element at row I and column J is true if there is an edge from I to J, otherwise false

Adjacency Matrix



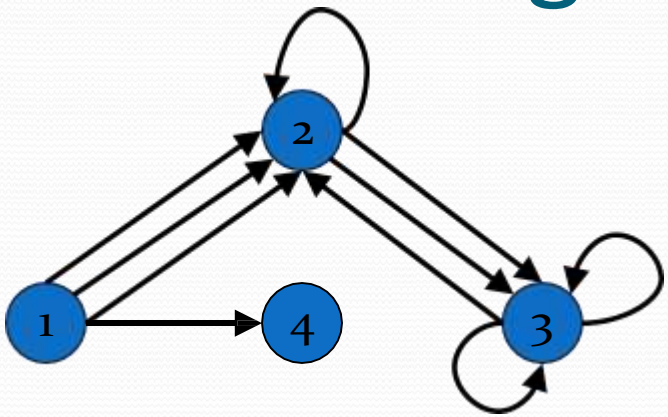
	0	1	2	3	4
0	false	false	true	false	false
1	false	false	false	true	false
2	false	true	false	false	true
3	false	false	false	false	false
4	false	false	false	true	false

Adjacency Matrix



	1	2	3	4	5
1	0	1	1	1	0
2	1	0	0	1	0
3	1	0	0	1	1
4	1	1	1	0	1
5	0	0	1	1	0

Adjacency Matrix
-Directed Multigraphs



A:

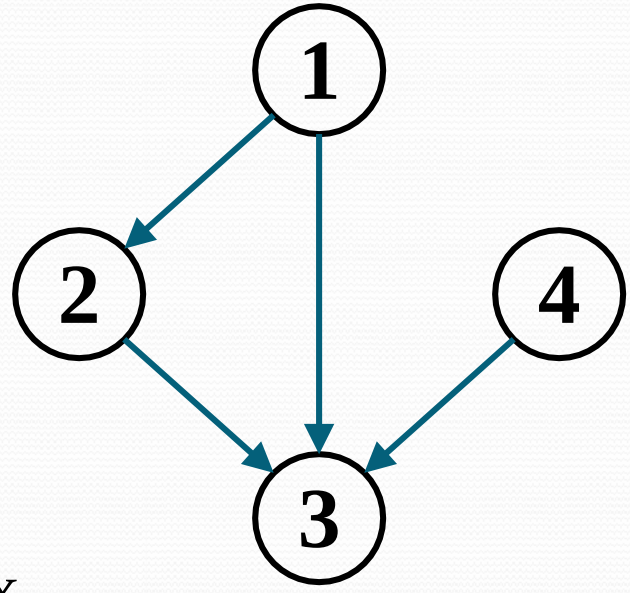
$$\begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Adjacency Lists Representation

- A graph of n nodes is represented by a one-dimensional array L of linked lists, where
  - L[i] is the linked list containing all the nodes adjacent from node i.
  - The nodes in the list L[i] are in no particular order

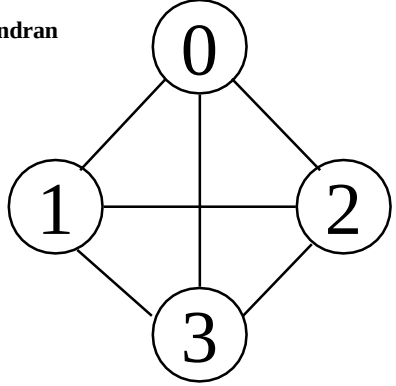
Graphs: Adjacency List

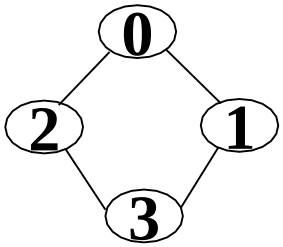
- Adjacency list: for each vertex $v \in V$, store a list of vertices adjacent to v
- Example:
  - Adj[1] = {2,3}
  - Adj[2] = {3}
  - Adj[3] = {}
  - Adj[4] = {3}
- Variation: can also keep a list of edges coming *into* vertex

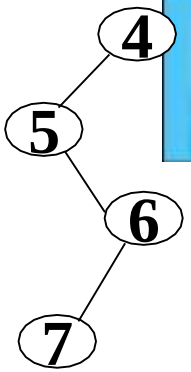


Graphs: Adjacency List

- How much storage is required?
  - The *degree* of a vertex v = # incident edges
    - Directed graphs have in-degree, out-degree
  - For directed graphs, # of items in adjacency lists is
    $$\Sigma \text{ out-degree}(v) = |E|$$
    For undirected graphs, # items in adjacency lists is
    $$\Sigma \text{ degree}(v) = 2\,|E|$$
- So: Adjacency lists take O(V+E) storage







0	1	2	3
1	0	2	3
2	0	1	3
3	0	1	2

G_1

0	1	
1	0	2
2		

G_3

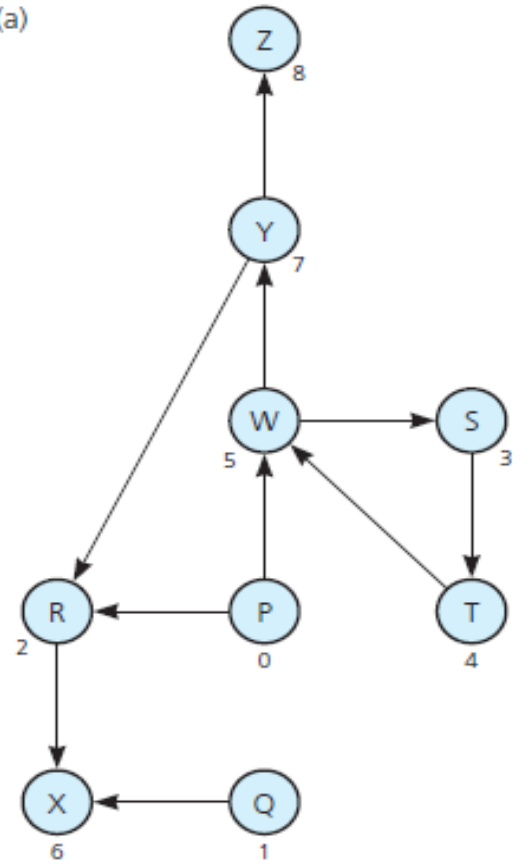
0	1	2
1	0	3
2	0	3
3	1	2
4	5	
5	4	6
6	5	7
7	6	

G_4

An undirected graph with n vertices and e edges ==> n head nodes and 2e list nodes

Implementing Graphs

(a)



(b)

		0	1	2	3	4	5	6	7	8
		P	Q	R	S	T	W	X	Y	Z
0	P	0	0	1	0	0	1	0	0	0
1	Q	0	0	0	0	0	0	1	0	0
2	R	0	0	0	0	0	0	1	0	0
3	S	0	0	0	0	1	0	0	0	0
4	T	0	0	0	0	0	1	0	0	0
5	W	0	0	0	1	0	0	0	1	0
6	X	0	0	0	0	0	0	0	0	0
7	Y	0	0	1	0	0	0	0	0	1
8	Z	0	0	0	0	0	0	0	0	0

ANY QUERY??