FUZZY LOGIC

What is Fuzzy Logic

- Fuzzy logic is a mathematical language to express something.
  - This means it has grammar, syntax, semantic like a language for communication.
- There are some other mathematical languages also known
  - **Relational algebra** (operations on sets)
  - **Boolean algebra** (operations on Boolean variables)
  - **Predicate algebra** (operations on well formed formulae (wff), also called predicate propositions)
- **Fuzzy logic deals with Fuzzy set or Fuzzy algebra.**

What is Fuzzy

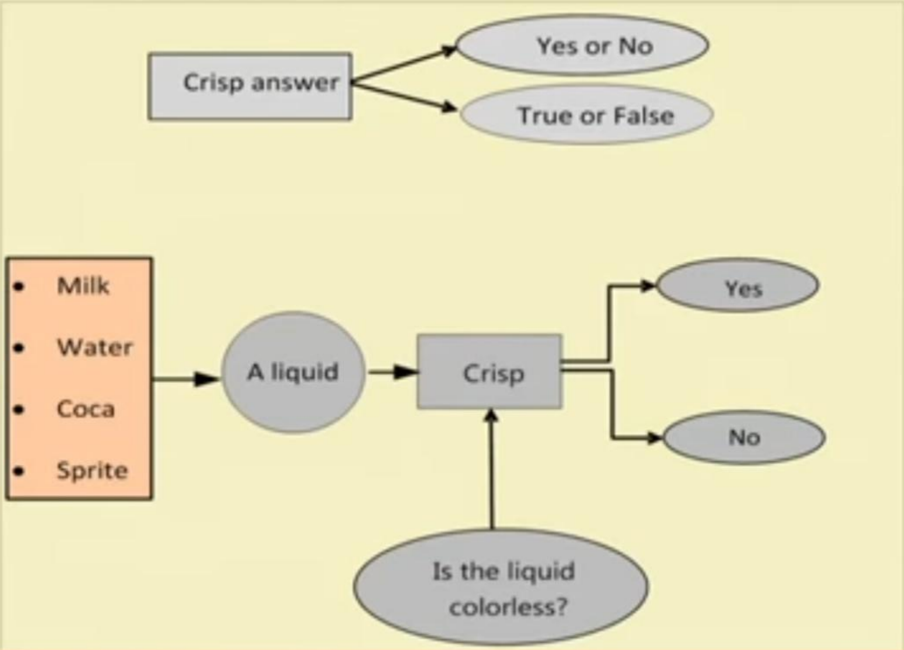
- Dictionary meaning of **fuzzy** is not clear, noisy, etc.

  Example: Is the picture on this slide is fuzzy?

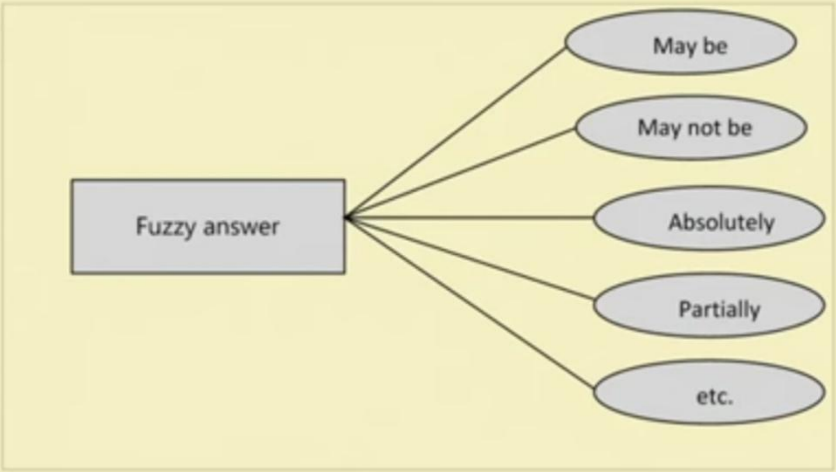
- Antonym of fuzzy is **crisp**

  Example: Are the chips crisp?

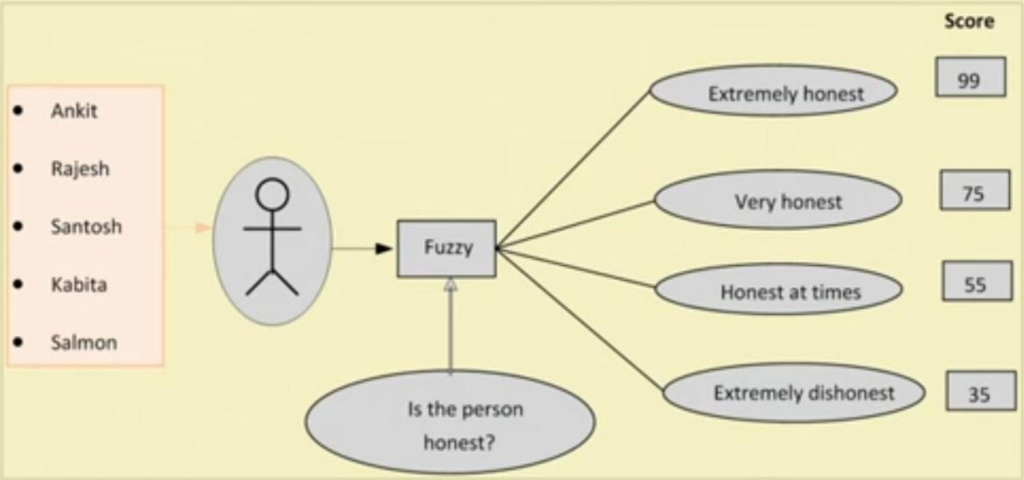
Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic

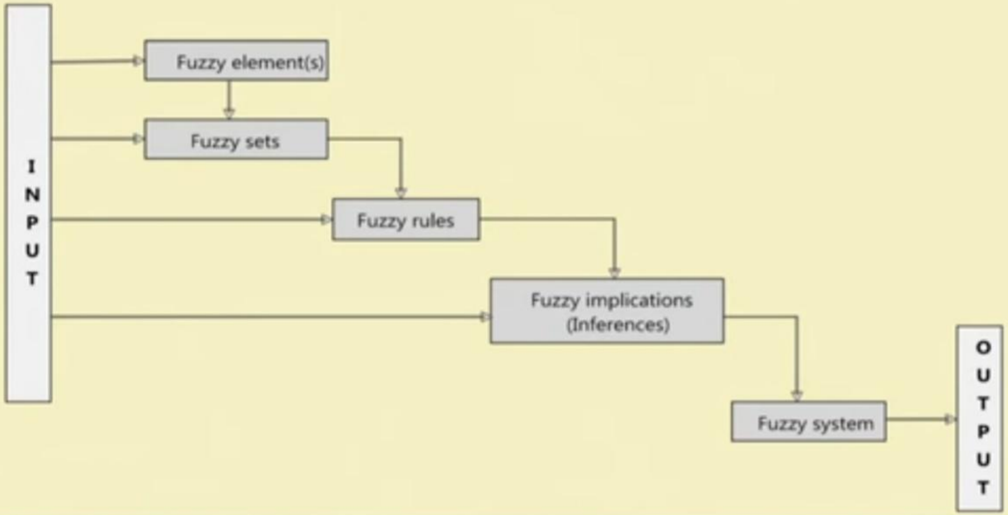


Example : Fuzzy logic vs. Crisp logic



World is fuzzy!

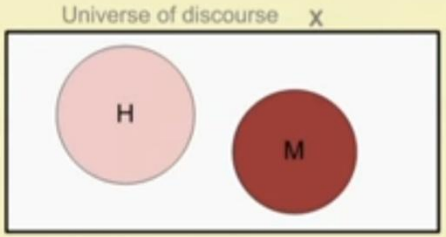
Concept of fuzzy system



Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

X = The entire population of India.
H = All Hindu population = $\{h_1, h_2, h_3, \ldots \ldots, h_L,\}$
M = All Muslim population =$\{m_1, m_2, m_3, \ldots \ldots, m_N,\}$



Here, All are the sets of finite numbers of individuals. Such a set is called crisp set.

Example of fuzzy set

Let us discuss about fuzzy set.

X = All students in NPTEL.

S = All **Good students**.

$S = \{(s, g(s)) \mid s \in X\}$ and $g(s)$ is a measurement of goodness of the student s.

Example:

$S = \{(Rajat, 0.8), (Kabita, 0.7), (Salman, 0.1), (Ankit, 0.9)\}$, etc.

Fuzzy set vs. Crisp set

Crisp set	Fuzzy set
▪ $S = \{s \mid s \in X\}$	▪ $F = (s, \mu(s)) \mid s \in X$ and $\mu(s)$ is the degree of s.
▪ It is a collection of elements.	▪ It is a collection of ordered pairs.
▪ Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary yes or no.	▪ Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of membership.

Fuzzy set vs. Crisp set

Note: A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:

$$H = \{(h_1, 1), (h_2, 1) \dots\dots\dots, (h_L, 1)\}$$

$$\text{Person} = \{(p_1, 0), (p_2, 0) \dots\dots\dots, (p_N, 0)\}$$

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

Degree of membership

How to decide the degree of memberships of elements in a fuzzy set?

City	Bangalore	Bombay	Hyderabad	Kharagpur	Madras	Delhi
μ	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of comfort can be judged?

Example: Course evaluation in a crisp way

$EX:$ Marks ≥ 90

$A:$ $80 \leq$ Marks < 90

$B:$ $70 \leq$ Marks < 80

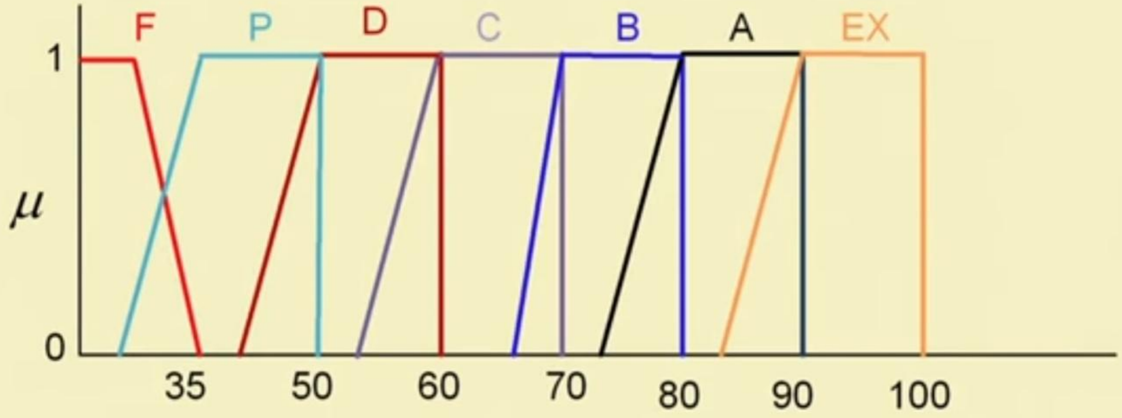
$C:$ $60 \leq$ Marks < 70

$D:$ $50 \leq$ Marks < 60

$P:$ $35 \leq$ Marks < 50

$F:$ Marks ≤ 35

Example: Course evaluation in a fuzzy way



Few examples of fuzzy set

- High Temperature
- Low Pressure
- Colour of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range [0...1].

Some basic terminologies and notations

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and $x \in X$, then a fuzzy set A in X is defined as a set of ordered pairs, that is
$A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A(x)$ is called the membership function for the fuzzy set A.

Note: $\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Question: How (and who) decides $\mu_A(x)$ for a fuzzy set A in X?

Some basic terminologies and notations

Example:

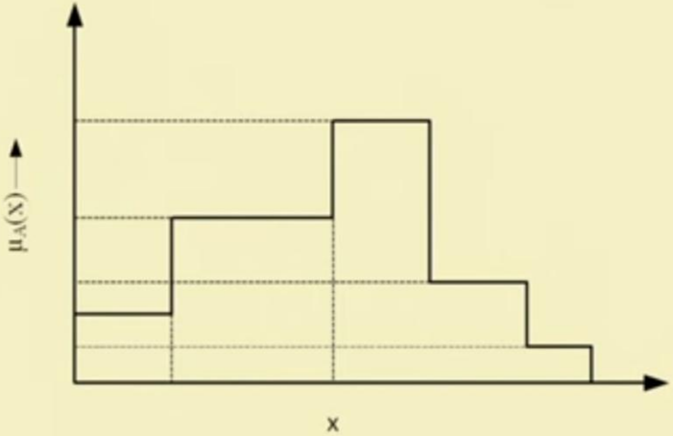
X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

Membership function with discrete membership values

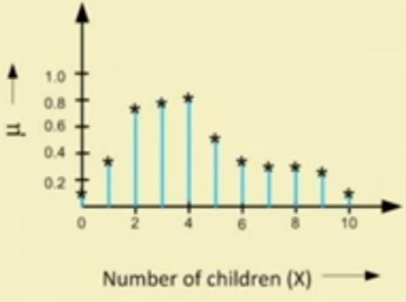
The membership values may be of discrete values.



A fuzzy set with discrete values of μ

Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



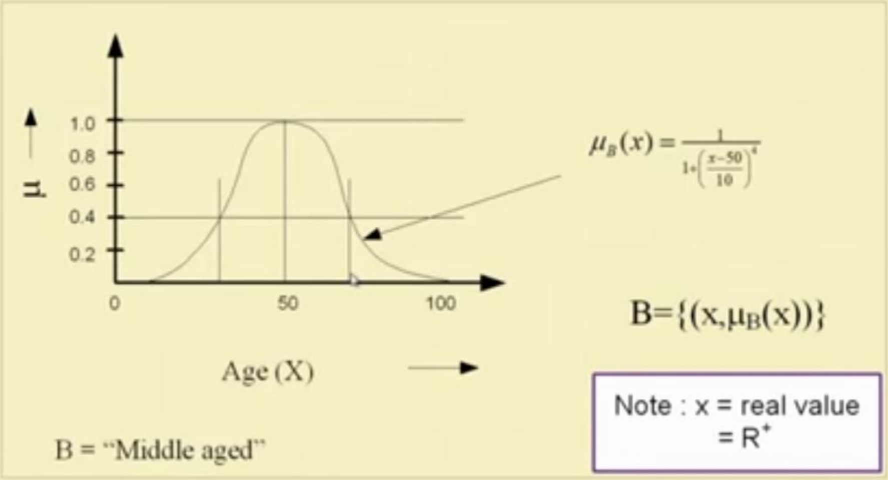
A = "Happy family"

A ={(0,0.1),(1,0.30),(2,0.78)......(10,0.1)}

Note : X = discrete value

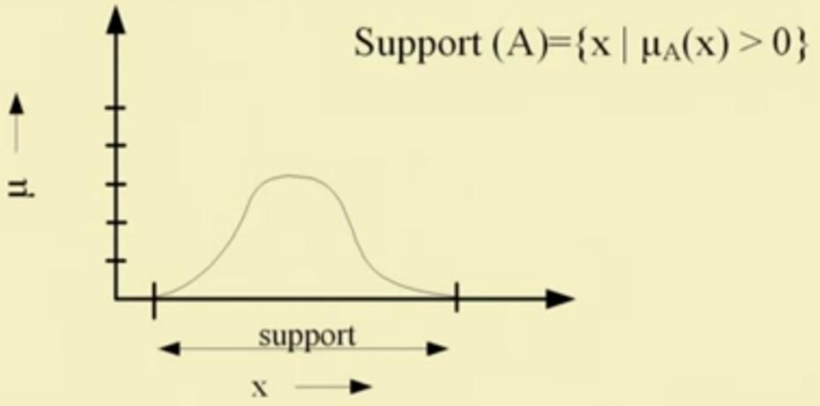
How you measure happiness ??

Membership function with continuous membership values



Fuzzy terminologies: Support

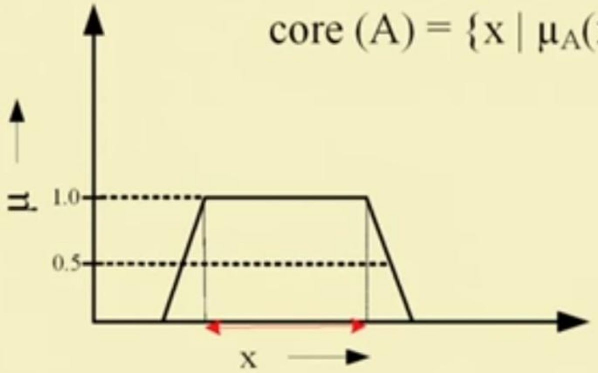
Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(X) > 0$



Fuzzy terminologies: Core

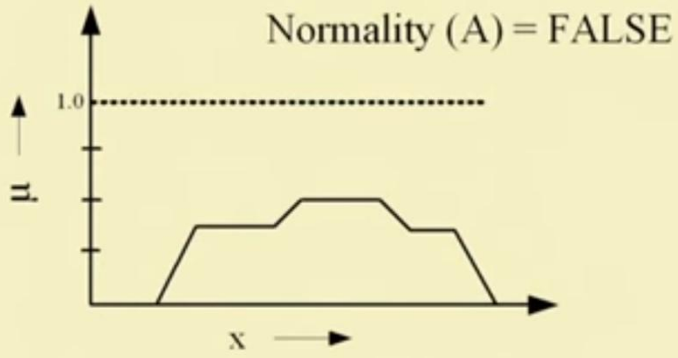
Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(X) = 1$

$$\text{core}(A) = \{x \mid \mu_A(x) = 1\}$$



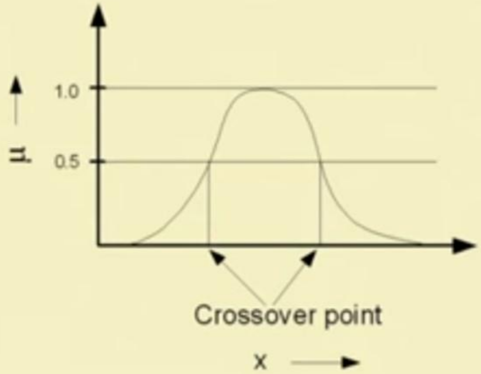
Fuzzy terminologies: Normality

Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(X) = 1$



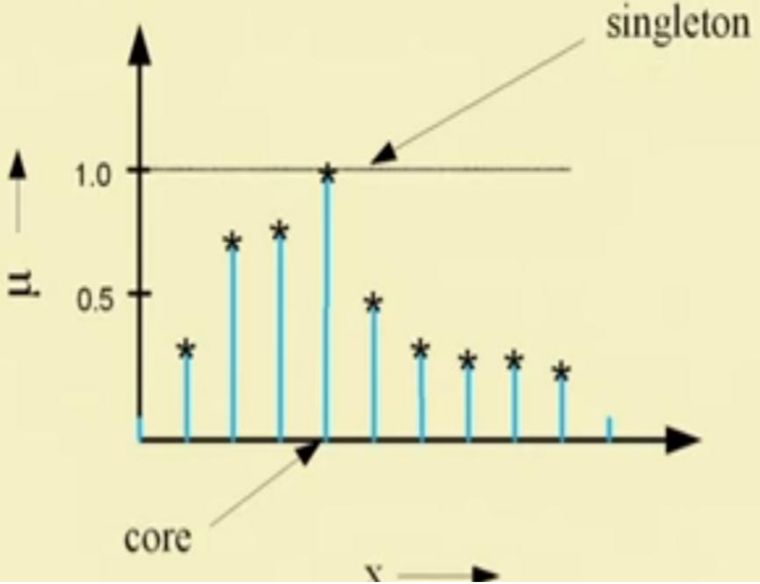
Fuzzy terminologies: Crossover points

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(X) = 0.5$. That is Crossover (A) $=\{x|\mu_A(x) = 0.5\}$



Fuzzy terminologies: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = \{x|\mu_A(x) = 1\}$



Fuzzy terminologies: α-cut and strong α-cut

α-cut and strong α-cut :

- ✓ The α-cut of a fuzzy set A is a crisp set defined by

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}$$

- ✓ Strong α-cut is defined similarly :

$$A'_\alpha = \{x | \mu_A(x) > \alpha\}$$

Note : Support (A) = A_0' and Core (A) = A_1.

Fuzzy terminologies: Bandwidth

Bandwidth :

For a fuzzy set, the bandwidth (or width) is defined as the distance between the two unique crossover points:

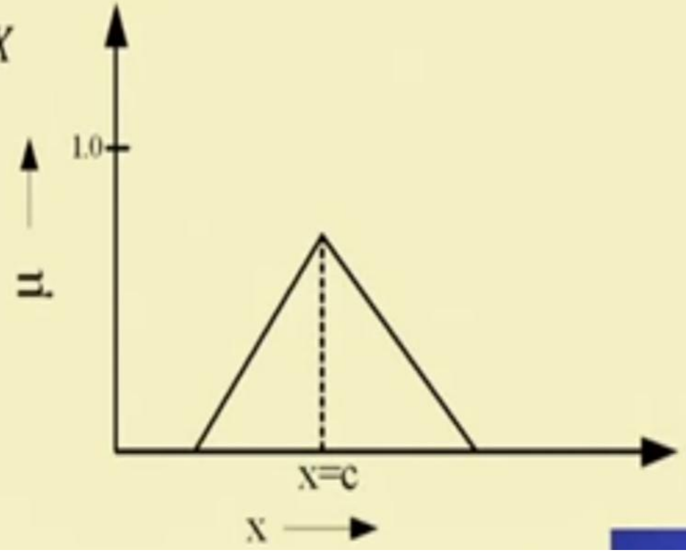
$$\text{Bandwidth}\,(A) = |x_1 - x_2|$$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Fuzzy terminologies: Symmetry

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in X$



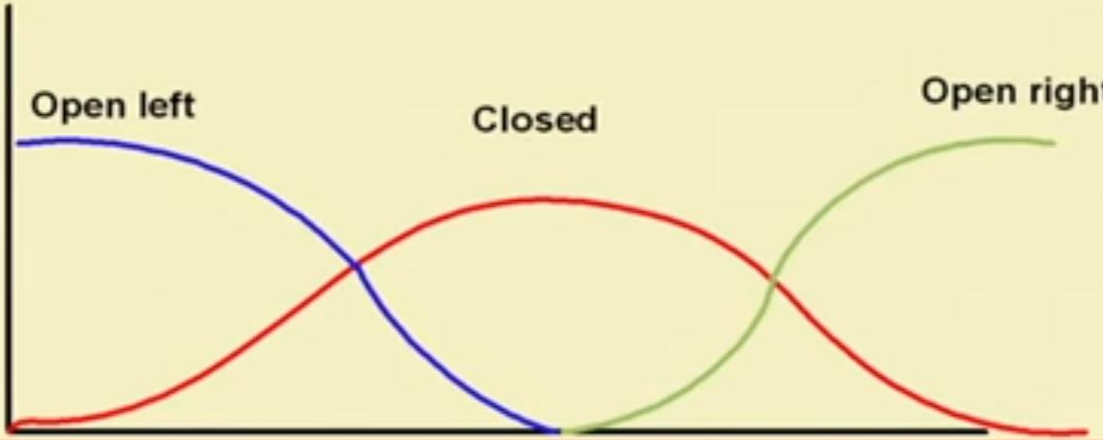
Fuzzy terminologies: Open and Closed

A fuzzy set A is

Open left : If $\lim x \to -\infty\ \mu_A(x) = 1$ and $\lim x \to +\infty\ \mu_A(x) = 0$

Open right: If $\lim x \to -\infty\ \mu_A(x) = 0$ and $\lim x \to +\infty\ \mu_A(x) = 1$

Closed: If $\lim x \to -\infty\ \mu_A(x) = \lim x \to +\infty\ \mu_A(x) = 0$



Fuzzy vs. Probability

Fuzzy : When we say about certainty of a thing

Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed.

Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

Probability: When we say about the chance of an event to occur

Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

Prediction vs. Forecasting

The Fuzzy vs. Probability is analogical to Prediction vs. Forecasting

Prediction : When you start guessing about things.

Forecasting : When you take the information from the past job and apply it to new job.

The main difference:

Prediction is based on the best guess from experiences.

Forecasting is based on data you have actually recorded and packed from previous job.

FUZZY LOGIC

Lecture-3

Fuzzy membership functions

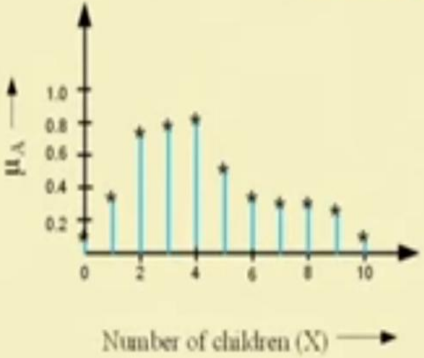
A fuzzy set is completely characterized by its membership function (sometimes abbreviated as MF and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

Note: A membership function can be on

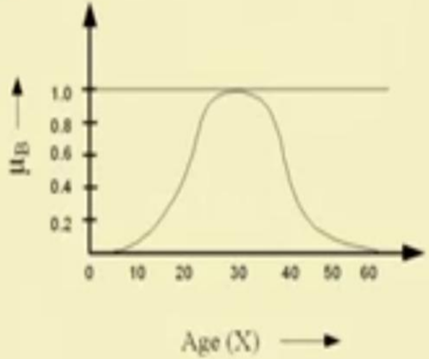
a) a discrete universe of discourse and

b) a continuous universe of discourse.

Example:



A = Fuzzy set of "Happy family"

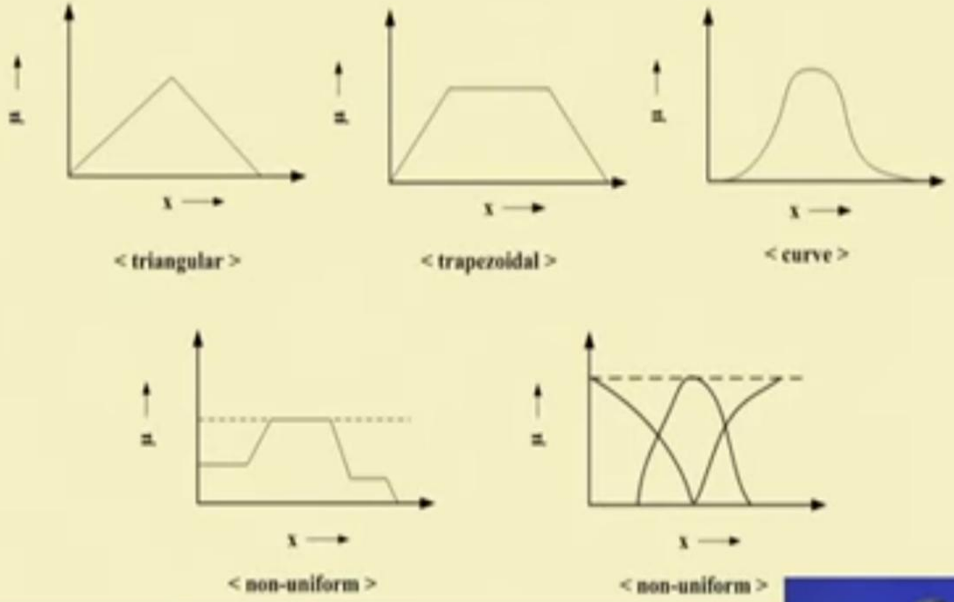


B = "Young age"

Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

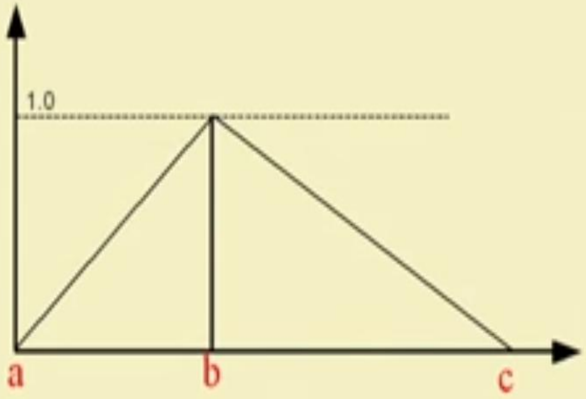
Following figures show some typical examples of membership functions.



Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

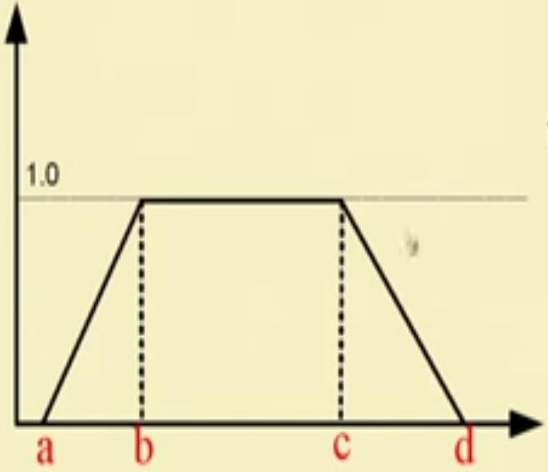
Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.



$$\text{triangle}\,(x; a, b, c) = \begin{cases} 0 & if\ x \le a \\ \dfrac{x-a}{b-a} & if\ a \le x \le b \\ \dfrac{c-x}{c-b} & if\ b \le x \le c \\ 0 & if\ c \le x \end{cases}$$

Fuzzy MFs: Trapezoidal

A **trapezoidal MF** is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

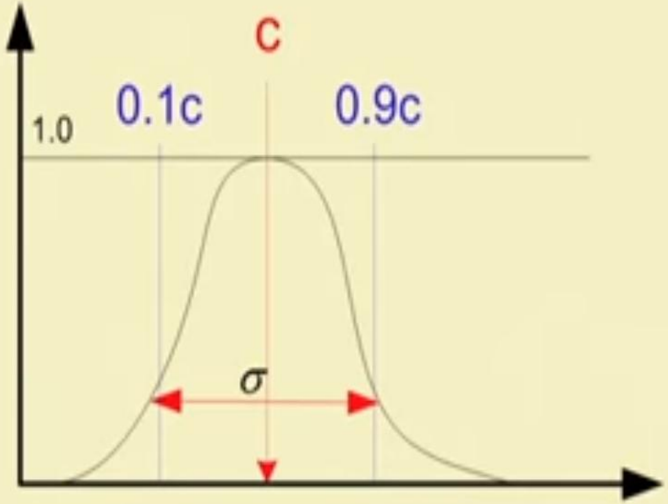


$$\text{trapozoid}\,(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b \le x \le c \\ \dfrac{d-x}{d-c} & \text{if } c \le x \le d \\ 0 & \text{if } d \le x \end{cases}$$

Fuzzy MFs: Gaussian

A **Gaussian MF** is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

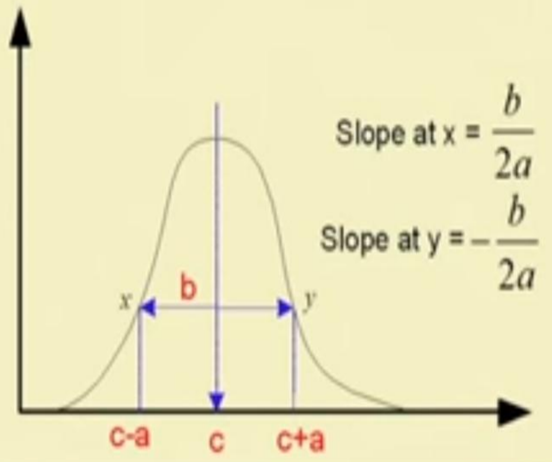
$$gaussian\,(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



Fuzzy MFs: Generalized bell

It is also called Cauchy MF. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

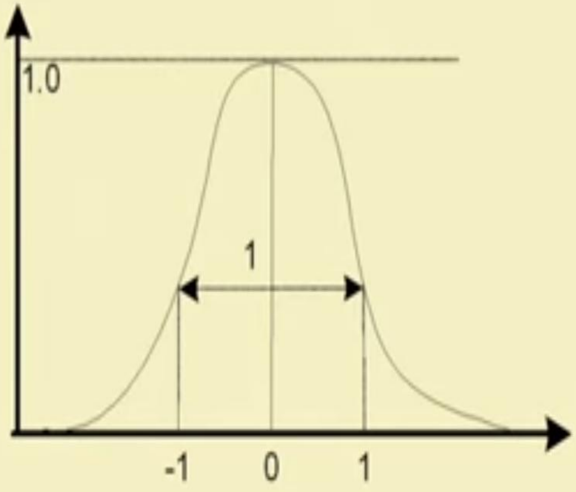
$$bell(x;\ a, b, c) = \frac{1}{1 + \left|\frac{x - c}{a}\right|^{2b}}$$



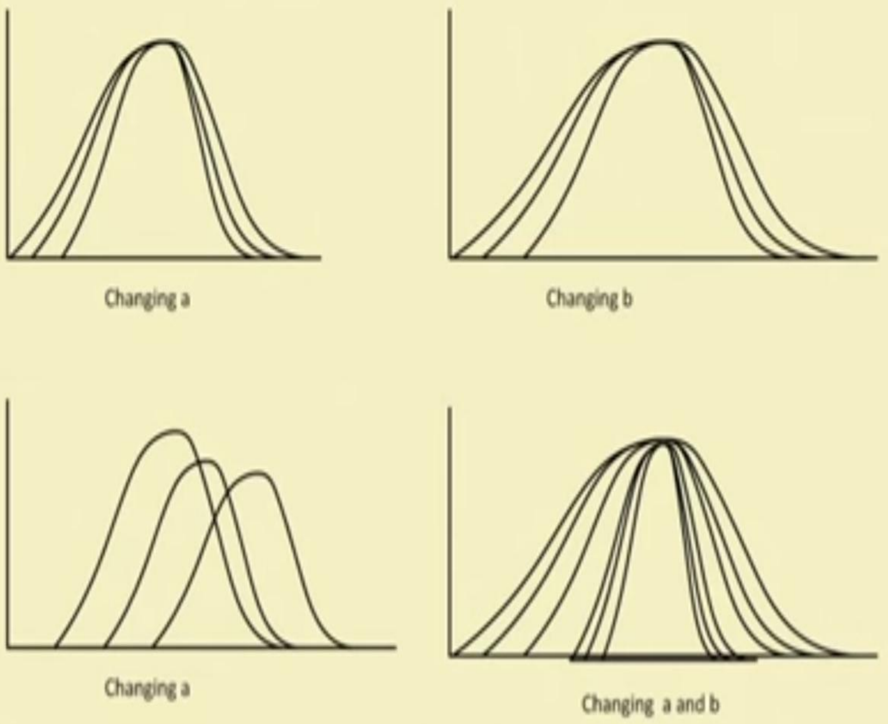
Example: Generalized bell MFs

Example: $\mu(x) = \frac{1}{1+|x|^2}$;

a = b = 1 and c = 0;

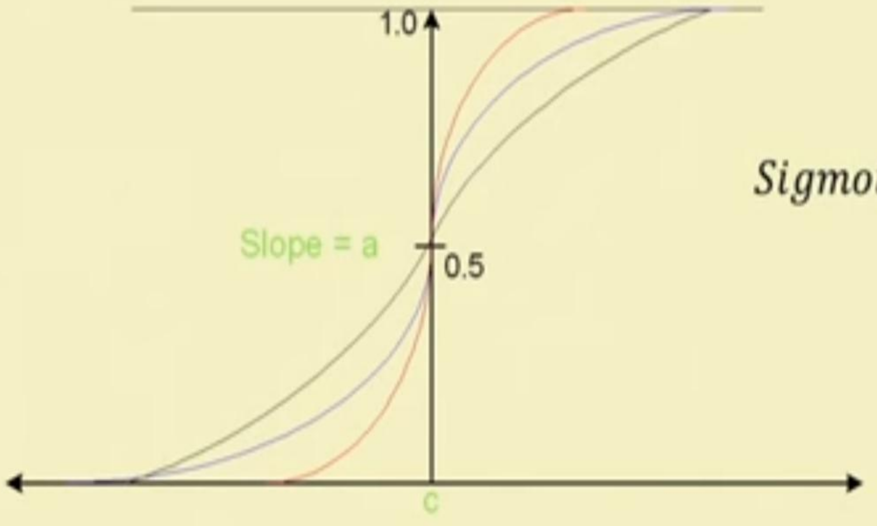


Generalized bell MFs: Different shapes



Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c;



$$Sigmoid(x;\, a, c) = \frac{1}{1 + e^{-\left[\frac{x-c}{a}\right]}}$$

Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks ≤ 90

Very good = $75 \leq$ Marks ≤ 90

Good = $60 \leq$ Marks ≤ 75

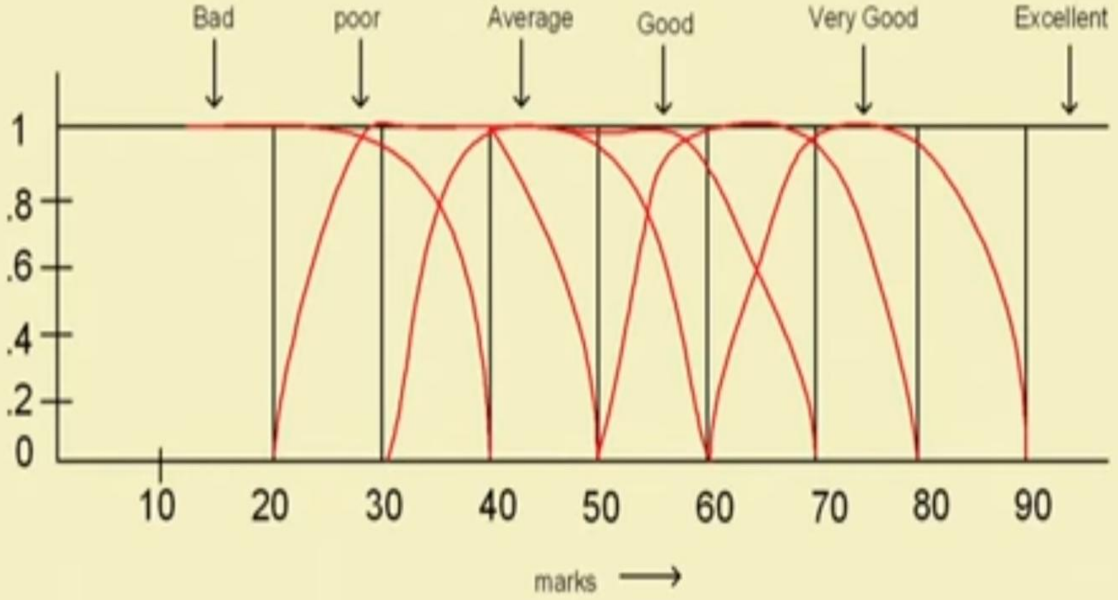
Average = $50 \leq$ Marks ≤ 60

Poor = $35 \leq$ Marks ≤ 50

Bad= Marks ≤ 35

Grading System

A fuzzy implementation will look like the following.



You can decide a standard fuzzy MF for each of the fuzzy grade.

Few More on Membership Functions

Generation of MFs

Given a membership function of a fuzzy set representing a linguistic hedge, we can derive many more MFs representing several other linguistic hedges using the concept of Concentration and Dilation.

1. **Concentration:** $A^k = [\mu_A(x)]^k; k > 1$
2. **Dilation:** $A^k = [\mu_A(x)]^k; k < 1$

Example : Age = { Young, Middle-aged, Old }

Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.
Similarly, with Old we can have : Not old, Very old, Very very old, Extremely old, etc.

Thus, $\mu_{Extremely\ old}(x) = (((\mu_{Old}(x))^2)^2)^2$ and so on

Or, $\mu_{More\ or\ less\ old}(x) = A^{0.5} = (\mu_{Old}(x))^{0.5}$

Linguistic variables and values

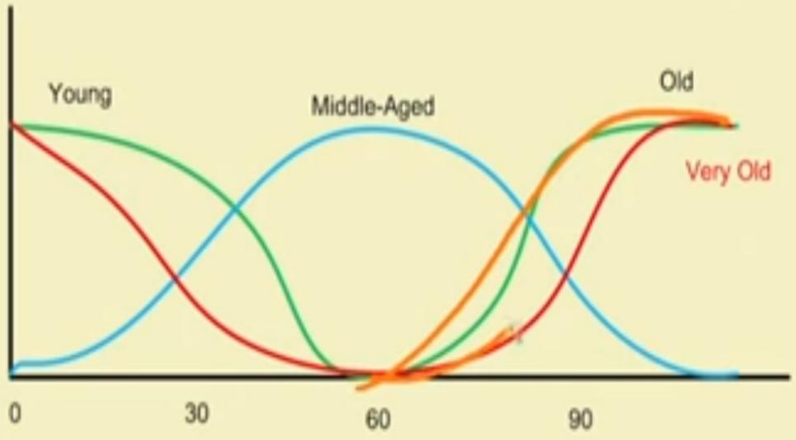
$\mu_{young}(x)$ =bell(x,20,2,0) = $\frac{1}{1+(\frac{x}{20})^4}$

$\mu_{old}(x)$ =bell(x,30,3,100) = $\frac{1}{1+(\frac{x-100}{30})^6}$

$\mu_{middle-aged}(x)$ =bell(x,30,60,50)

Not young=$\overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$

Young but not too young = $\mu_{young}(x) \cap \overline{\mu_{young}(x)}$



FUZZY LOGIC

Lecture-4

Introduction to Soft Computing

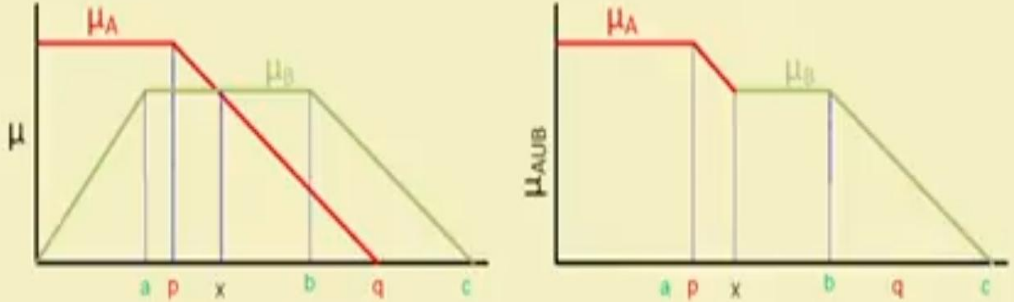
Operations on Fuzzy sets

Basic fuzzy set operations: Union

Union (A ∪ B): $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$



Basic fuzzy set operations: Intersection

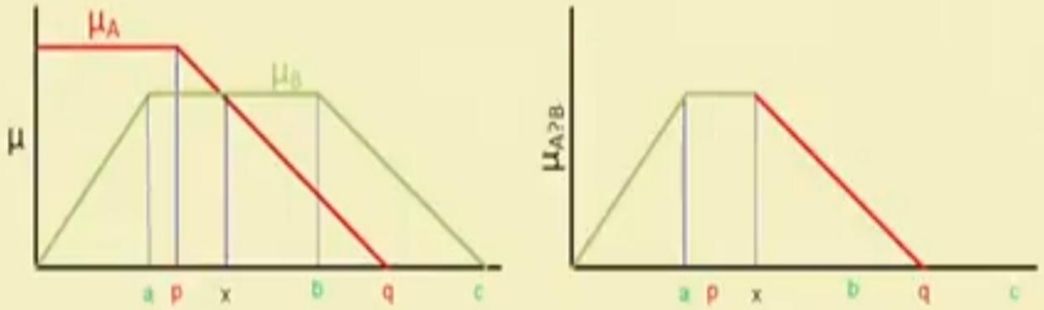
Intersection (A ∩ B): $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$



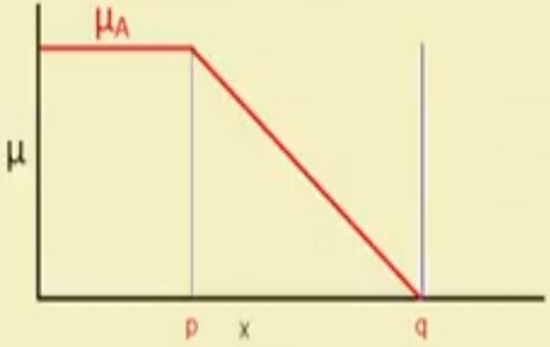
Basic fuzzy set operations: Complement

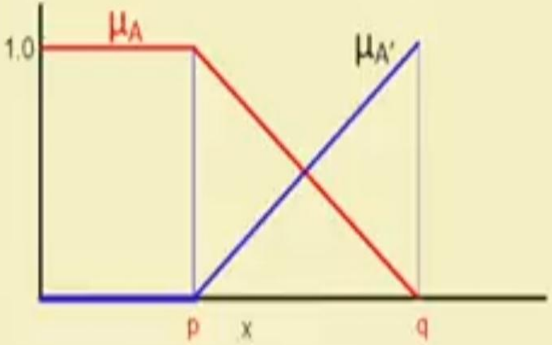
Complement (A^c): $\mu_{A^c}(x) = 1 - \mu_A(x)$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^c = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$





Basic fuzzy set operations: Products

Algebric product or Vector product ($A \cdot B$):

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Scalar product ($\alpha \times A$):

$$\mu_{\alpha A}(x) = \alpha \times \mu_A(x)$$

Basic fuzzy set operations: Sum and Difference

Sum (A + B):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)$$

Difference ($A - B = A \cap B^C$)**:**

$$\mu_{A-B}(x) = \mu_{A\cap B^C}(x)$$

Disjunctive sum:

$$A \oplus B = (A^C \cap B) \cup (A \cap B^C)$$

Bounded Sum:

$$| A(x) \oplus B(x) | = \mu_{|A(x)\oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference:

$$| A(x) \ominus B(x) | = \mu_{|A(x)\ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Basic fuzzy set operations: Equality and Power

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α:

$$\mu_{A^\alpha}(x) = (\mu_A(x))^\alpha$$

- ✓ If $\alpha < 1$, then it is called dilation
- ✓ If $\alpha > 1$, then it is called concentration

Basic fuzzy set operations: Cartesian product

Caretsian Product ($A \times B$): $\mu_{A\times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Example:

$A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$

$B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$\mathrm{A} \times B = \min\big(\mu_A(x), \mu_B(y)\big) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Properties of fuzzy sets

Commutativity :

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of fuzzy sets

Idempotence :

$$A \cup A = A$$
$$A \cap A = \emptyset;$$
$$A \cup \emptyset; = A$$
$$A \cap \emptyset; = \emptyset;$$

Transitivity :

If $A \subseteq B; B \subseteq C$ then $A \subseteq C$

Involution :

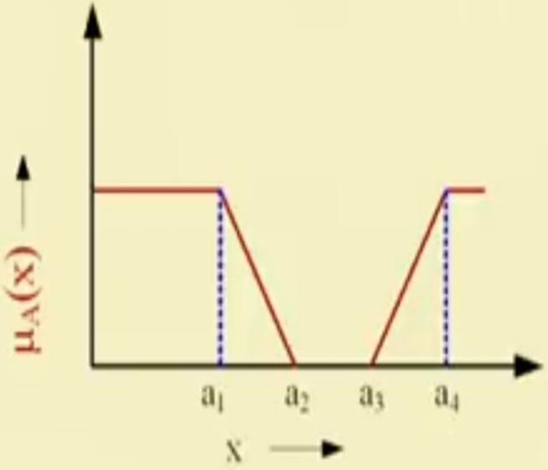
$$(A^c)^c = A$$

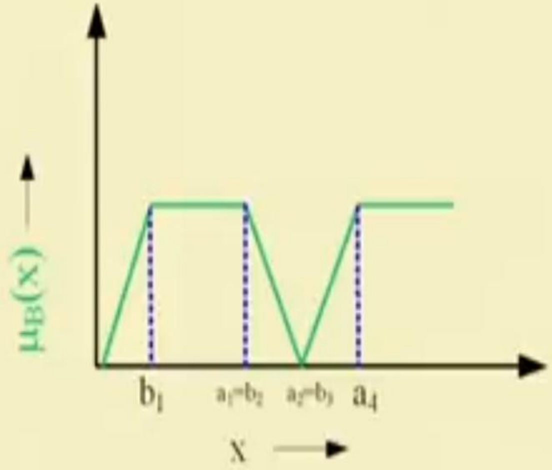
De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Example 1: Fuzzy Set Operations

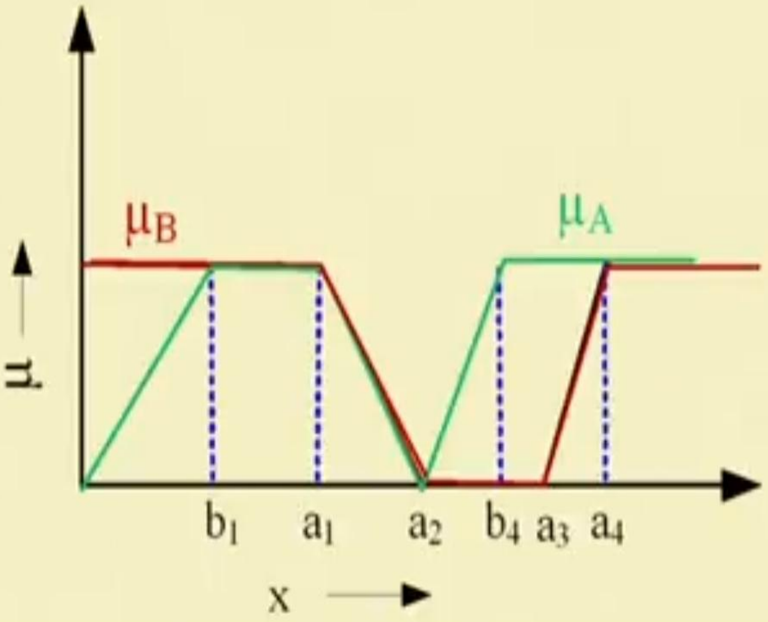
Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.





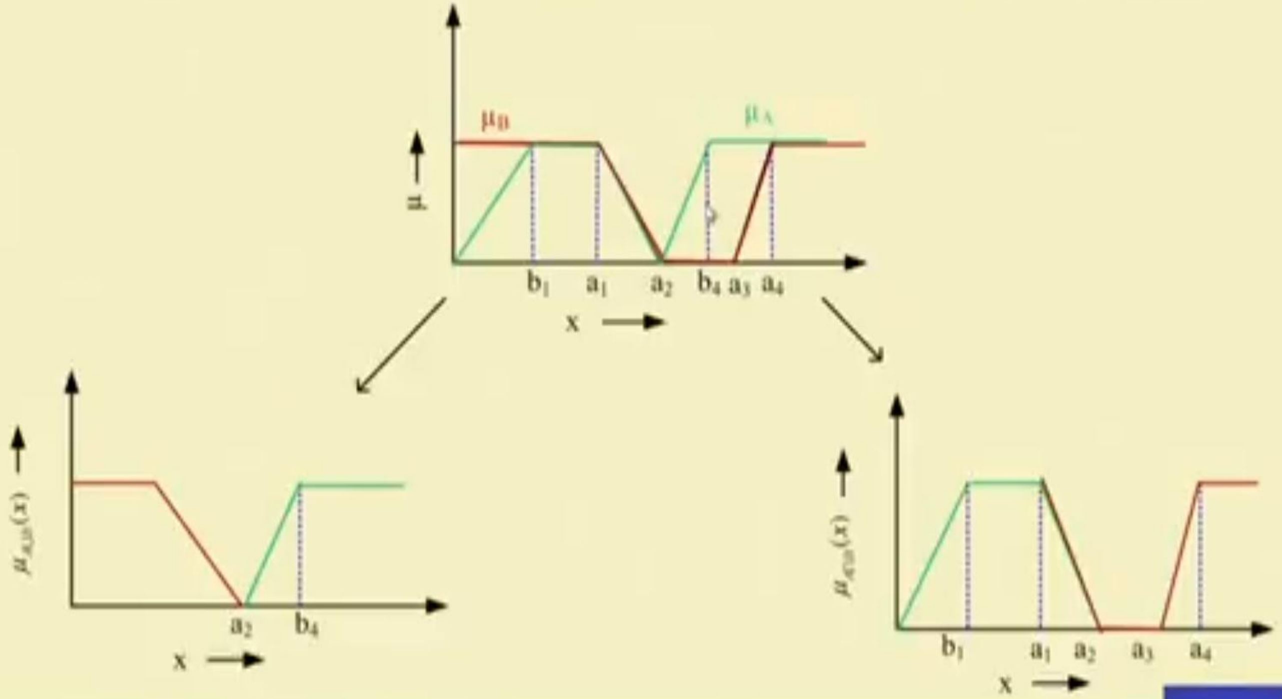
Example 1: Plotting two sets on the same graph

Let's plot the two membership functions on the same graph



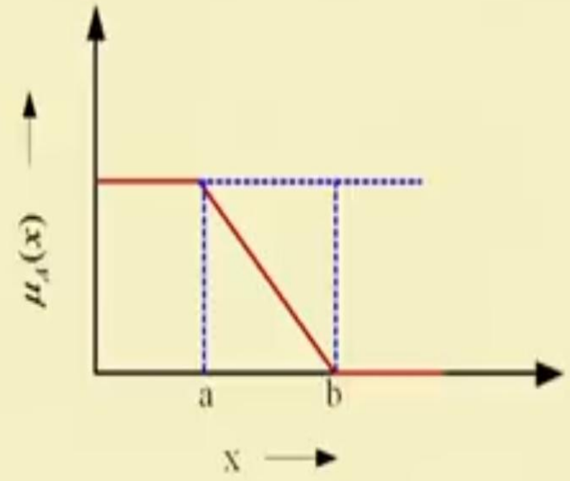
Example 1: Union and Intersection

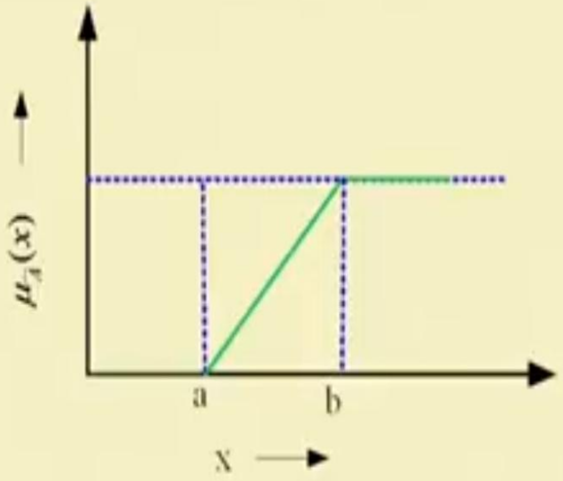
The plots of union $A \cup B$ and intersection $A \cap B$ are shown in the following.



Example 1: Complementation

The plots of union $\mu_{\bar{A}}(x)$ of the fuzzy set A is shown in the following.





Fuzzy set operations: Practice

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

I. $\bar{A}, \bar{B}$
II. $A \cup B$
III. $A \cap B$
IV. $(A \cup B)^c$

[Hint: Use De' Morgan law]

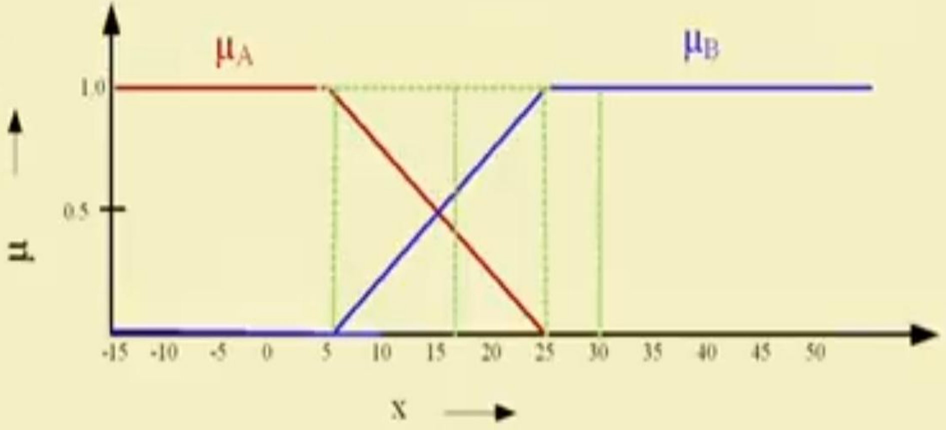
Example 2: A real-life example

Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

A = **Cold climate** with $\mu_A(x)$ as the MF.

B = **Hot climate** with $\mu_B(x)$ as the M.F.

Here, X being the universe of discourse representing entire range of temperatures.



Example 2: A real-life example

What are the fuzzy sets representing the following?

1. Not cold climate
2. Not hot climate
3. Extreme climate
4. Pleasant climate

Note: Note that "Not cold climate" ≠ "Hot climate" and vice-versa.

Example 2: A real-life example

Answer would be the following.

✓ **Not cold climate**

$\bar{A}$ with $1 - \mu_A(x)$ as the MF.

✓ **Not hot climate**

$\bar{B}$ with $1 - \mu_B(x)$ as the MF.

✓ **Extreme climate**

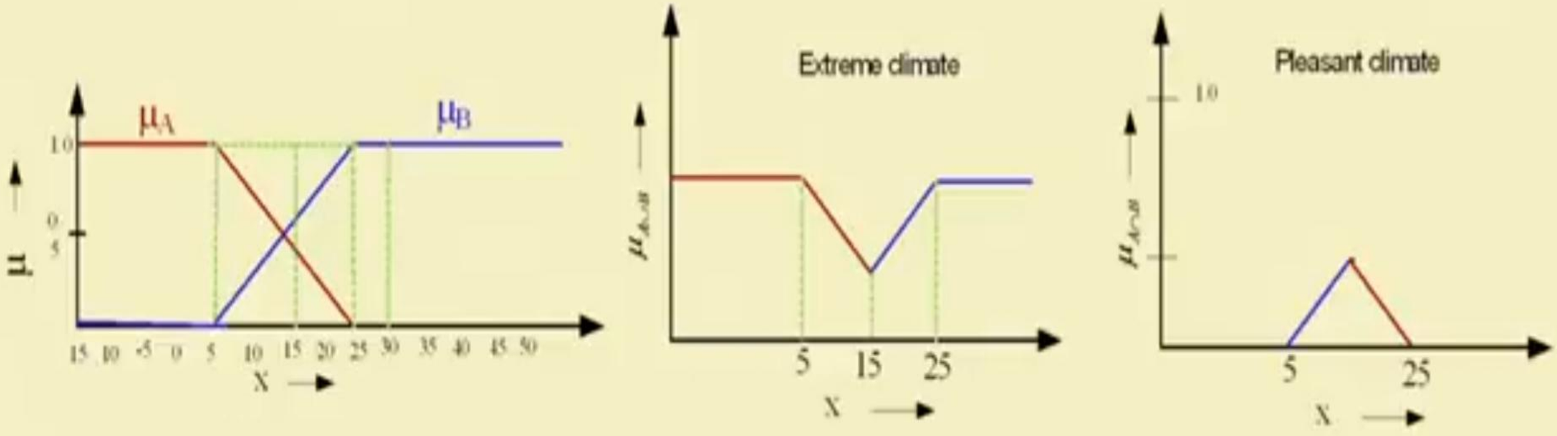
$A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.

✓ **Pleasant climate**

$A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

Example 2: A real-life example

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



THANKS