Fuzzy Relations

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Crisp relations

- **Order pairs:**

  Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

  $$A \times B = \{(a, b) | a \in A \; and \; b \in B\}$$

  Note :

  (1) $A \times B \neq B \times A$ (2) $|A \times B| = |A| \times |B|$

  (3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

  A particular mapping so mentioned is called a relation.

Crisp relations

Example:

Consider the two crisp sets A and B as given below.
$A = \{1, 2, 3, 4\}$ $B = \{3, 5, 7\}$.

Then, $A \times B = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7)\}$

Let us define a relation as $R = \{(a,b) | b = a + 1, (a,b) \in A \times B\}$

Then, $R = \{(2,3), (4,5)\}$ in this case.

Crisp relations

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

- **Union:** $R(x, y) \cup S(x, y) = max(R(x, y), S(x, y));$
- **Intersection:** $R(x, y) \cap S(x, y) = min(R(x, y), S(x, y));$
- **Complement:** $\overline{R(x, y)} = 1 - R(x, y)$

Example: Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the following

- $R \cup S$
- $R \cap S$
- $\bar{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z. Then, $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \; and \; (y, z) \in S \; and \; \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x, z) = max\{min\{R(x, y), S(y, z) \; and \; \forall y \in Y\}\}$$

Composition: Composition

Example : Given $X = \{1,3,5\}; Y = \{1,3,5\}; R = \{(x,y)|y = x + 2\};$
$S = \{(x,y)|x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have $R = \{(1,3),(3,5)\},\ S = \{(1,3),(1,5),(3,5)\}$

$$R = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition

$$R \circ S = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy relations

Example:

$X = \{typhoid, viral, cold\}, Y = \{running\ nose, high\ temp, shivering\}$

The fuzzy relation R is defined as

$$R = \begin{array}{c} typhoid \\ viral \\ cold \end{array} \begin{array}{c} \begin{array}{ccc} running\ nose & high\ temperature & shivering \end{array} \\ \begin{bmatrix} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Fuzzy Cartesian product

Suppose

- A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given

by $\mu_R(x, y) = \mu_{AxB}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Fuzzy Cartesian product

Example :

$A = \{(a1, 0.2), (a2, 0.7), (a3, 0.4)\}$ and $B = \{(b1, 0.5), (b2, 0.6)\}$

$$R = A \times B = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} \boldsymbol{b_1} & \boldsymbol{b_2} \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

- **Union:** $\mu_{R \cup S}(a, b) = max\{\mu_R(a, b), \mu_S(a, b)\}$
- **Intersection:** $\mu_{R \cap S}(a, b) = min\{\mu_R(a, b), \mu_S(a, b)\}$
- **Complement:** $\mu_{\bar{R}}(a, b) = 1 - \mu_R(a, b)$
- **Composition:** $T = R \circ S$

Operations on Fuzzy relations: Example

Example : $X = (x_1, x_2, x_3),\ Y = (y_1, y_2),\ Z = (z_1, z_2, z_3),$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \quad \text{and} \quad S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$$\mu_{R \circ S}(x_1, y_1) = max\{min(\mu_R(x_1, y_1), \mu_S(y_1, z_1)), min(\mu_R(x_1, y_2), \mu_S(y_2, z_1))\}$$

$$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5 \text{ and so on.}$$